INVARIANTS WITH RESPECT TO SPECIAL PROJECTIVE TRANSFORMATIONS

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CHAPTER I

INTRODUCTION

The possibility of transforming a curve with any singularities into a curve with multiple points with distinct tangents goes back to Kronecker, who communicated the theorem verbally to Riemann and Weierstrass in 1858.[1]

This paper proposes to treat certain phases of actually reducing singularities on surfaces by a detailed study of just which surfaces are invariant under a selected cyclic group of transformations (T, T^2, ... T^p; where $T^p = 1$) of prime order p. The generator T is then used in combination with certain quadratic transformations to reduce the singularity, according to the classical theory.

The quadratic reduction process has been set up for an I.B.M. (International Business Machine) calculator and the results used to make empirical observations which parallel somewhat the results of Godeaux[2] pertaining to singular points on a curve.

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[1] Emch, Arnold, "Analysis of Singularities of Plane Algebraic Curves," Bulletin of the National Research Council, no. 63, 1928, p. 59, lines 3-6.

[2] Godeaux, I., "Sur les Homographies Planes Cycliques," Memoires de la Societe Des Sciences de Liege, t. XV, 1930, pp. 1-26.

CHAPTER II

THE GENERAL INVARIANT POLYNOMIAL

In order to determine just which surfaces are invariant under the collineation

(1) $T;\ x_1':x_2':x_3':x_4' = x_1:Ex_2:E^2x_3:E^3x_4$ where $E^p = 1$

the intent is to exhibit a method for writing down all the terms of any homogeneous polynomial in x_1, x_2, x_3, and x_4 of prime degree $p > 3$ and invariant under T.

For this purpose consider the general term $x_1^a x_2^b x_3^c x_4^d$ which is defined as follows: let the integral exponents a, b, c, and d be members of the congruence classes[1] $\bar{n} = n + (p)$, $\overline{r - 2n} = r\text{-}2n+(p)$, $\overline{n\text{-}2r} = n\text{-}2r+(p)$, $\bar{r} = r+(p)$, respectively, and lie in the interval 0 to p-1 inclusive.

Terms of this form are of degree p or some multiple of p since

(2) $\bar{n} + \overline{r\text{-}2n} + \overline{n\text{-}2r} + \bar{r} \equiv \bar{0}$

Furthermore invariance is assured, since

(3) $x_1^a x_2^b x_3^c x_4^d \underset{T}{\sim} x_1^a E^b x_2^b E^{2c} x_3^c E^{3d} x_4^d$ and

(4) $\overline{r\text{-}2n} + 2\ \overline{n\text{-}2r} + 3\ \bar{r} \equiv \bar{0}$

Let us confine our attention to terms of degree p by the following agreement: for each integer r in the following intervals, let n assume all integral values

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[1]For notation see Jacobsen, Nathan, <u>Lectures in Abstract Algebra</u>, Vol. I , p. 66.

in its corresponding interval:

(A1) To $0 \leqslant r \leqslant \frac{p}{3}$ corresponds $2r \leqslant n \leqslant \frac{p+r}{2}$,

(A2) To $0 \leqslant r \leqslant \frac{p}{2}$ corresponds $0 \leqslant n \leqslant \frac{r}{2}$ and

(A3) to $\frac{p+1}{2} \leqslant r \leqslant \frac{2p}{3}$ corresponds $2r-p \leqslant n \leqslant \frac{r}{2}$.

An involution of period thirteen has been considered in a recent paper,[2] even though the invariant surface of degree thirteen was not written out. However, the table showing all of these 44 terms is given below.

TABLE 1. VALUES OF (n,r) FOR p=13

r = 0;	n = 0,1,2,3,4,5,6
r = 1;	n = 0, 2,3,4,5,6,7
r = 2;	n = 0,1, 4,5,6,7
r = 3;	n = 0,1, 6,7,8
r = 4;	n = 0,1,2, 8
r = 5;	n = 0,1,2,
r = 6;	n = 0,1,2,3
r = 7;	n = 1,2,3
r = 8;	n = 3,4

For each pair of the above values (r,n) there corresponds one and only one distinct term of the polynomial except for n = r = 0 which gives x_1^{13}, x_2^{13}, x_3^{13}, x_4^{13}. The polynomial thus obtained in our example is $a_1x_1^{13} + a_2x_2^{13} + a_3x_3^{13} + a_4x_4^{13} + a_5x_1^{1}x_2^{11}x_3^{1} + a_6x_1^{2}x_2^{9}x_3^{2} + a_7x_1^{3}x_2^{7}x_3^{3} + a_8x_1^{4}x_2^{5}x_3^{4} +$

[2]Hutcherson, W. R., "Fifth Order Neighborhood of an Involution of Period Thirteen," Bulletin of American Mathematical Society, Vol. 57, No. 6 (Nov. 1951), p. 484.

$$a_9x_1^5x_2^3x_3^5 + a_{10}x_1^6x_2^1x_3^6 + a_{11}x_1^0x_2^1x_3^{11}x_4^1 + a_{12}x_1^2x_2^{10}x_4^1 +$$
$$a_{13}x_1^3x_2^8x_3^1x_4^1 + a_{14}x_1^4x_2^6x_3^2x_4^1 + a_{15}x_1^5x_2^4x_3^3x_4^1 + a_{16}x_1^6x_2^2x_3^4x_4^1 +$$
$$a_{17}x_1^7x_3^5x_4^1 + a_{18}x_2^2x_3^9x_4^2 + a_{19}x_1^1x_3^{10}x_4^2 + a_{20}x_1^4x_2^7x_4^2 +$$
$$a_{21}x_1^5x_2^5x_3^1x_4^2 + a_{22}x_1^6x_2^3x_3^2x_4^2 + a_{23}x_1^7x_2^1x_3^3x_4^2 + a_{24}x_2^3x_3^7x_4^3 +$$
$$a_{25}x_1^1x_2^1x_3^8x_4^3 + a_{26}x_1^6x_2^4x_3^0x_4^3 + a_{27}x_1^7x_2^2x_3^1x_4^3 + a_{28}x_1^8x_3^2x_4^3 +$$
$$a_{29}x_1^0x_2^4x_3^5x_4^4 + a_{30}x_1^1x_2^2x_3^6x_4^4 + a_{31}x_1^2x_2^0x_3^7x_4^4 + a_{32}x_1^8x_2^1x_3^0x_4^4 +$$
$$a_{33}x_1^0x_2^5x_3^3x_4^5 + a_{34}x_1^1x_2^3x_3^4x_4^5 + a_{35}x_1^2x_2^1x_3^5x_4^5 + a_{36}x_1^0x_2^6x_3^1x_4^6 +$$
$$a_{37}x_1^1x_2^4x_3^2x_4^6 + a_{38}x_1^2x_2^2x_3^3x_4^6 + a_{39}x_1^3x_2^0x_3^4x_4^6 + a_{40}x_1^1x_2^5x_3^0x_4^7 +$$
$$a_{41}x_1^2x_2^3x_3^1x_4^7 + a_{42}x_1^3x_2^1x_3^2x_4^7 + a_{43}x_1^3x_2^2x_3^0x_4^8 + a_{44}x_1^4x_2^0x_3^1x_4^8.$$

When $p = 11$, the surface[3] with 33 terms has been used in the projecting into a space of 32 dimensions certain invariant curves of order 44 on this surface. It has become evident through this study, however, that the representation of the above surface must have the one extra term, $a_{34}x_2x_3^9x_4$, as part of itself in order to represent the most general surface.

The case where $p = 41$ with 324 terms has been worked out also, and may be derived from Table 2. (See Appendix A.)

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[3]Hutcherson, W. R., "A Cyclic Involution of Period Eleven," Canadian Journal of Mathematics, Vol. III, No 2 (1951), p. 155, 158.

TABLE 2. VALUES OF (n,r) FOR p = 41.

r	n
r = 0;	n = 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20
r = 1;	n = 0, 2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21
r = 2;	n = 0,1, 4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21
r = 3;	n = 0,1, 6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22
r = 4;	n = 0,1,2, 8,9,10,11,12,13,14,15,16,17,18,19,20,21,22
r = 5;	n = 0,1,2, 10,11,12,13,14,15,16,17,18,19,20,21,22,23
r = 6;	n = 0,1,2,3, 12,13,14,15,16,17,18,19,20,21,22,23
r = 7;	n = 0,1,2,3, 14,15,16,17,18,19,20,21,22,23,24
r = 8;	n = 0,1,2,3,4, 16,17,18,19,20,21,22,23,24
r = 9;	n = 0,1,2,3,4, 18,19,20,21,22,23,24,25
r = 10;	n = 0,1,2,3,4,5, 20,21,22,23,24,25
r = 11;	n = 0,1,2,3,4,5, 22,23,24,25,26
r = 12;	n = 0,1,2,3,4,5,6, 24,25,26
r = 13;	n = 0,1,2,3,4,5,6, 26,27
r = 14;	n = 0,1,2,3,4,5,6,7
r = 15;	n = 0,1,2,3,4,5,6,7
r = 16;	n = 0,1,2,3,4,5,6,7,8
r = 17;	n = 0,1,2,3,4,5,6,7,8
r = 18;	n = 0,1,2,3,4,5,6,7,8,9
r = 19;	n = 0,1,2,3,4,5,6,7,8,9
r = 20;	n = 0,1,2,3,4,5,6,7,8,9,10
r = 21;	n = 1,2,3,4,5,6,7,8,9,10
r = 22;	n = 3,4,5,6,7,8,9,10,11
r = 23;	n = 5,6,7,8,9,10,11
r = 24;	n = 7,8,9,10,11,12
r = 25;	n = 9,10,11,12
r = 26;	n = 11,12,13
r = 27;	n = 13

The limits in (A) were suggested, by a study of Table 2. However, once they are obtained, it is a simple, although detailed procedure to check all these limits. For example, $r = 0$ and $n = 2r = 0$ are paired. Hence, $n-2r$ and $r-2n = 0$. In this case $x_1^0 x_2^0 x_3^0 x_4^0$ is the corresponding term. Let it be understood that this exceptional term shall represent the four terms x_1^p, x_2^p, x_3^p, and x_4^p. It will be convenient to let its occurrence in (A1) represent x_2^p and in (A2) represent x_3^p.

When $r = 0$ and $n = \frac{p+r-1}{2} = \frac{p-1}{2}$, then $r-2n = -p+1$, and $n-2r = \frac{p-1}{2}$. Replacing $-p+1$ by 1, according to the previous agreement on exponents, the corresponding term, $x_1^{\frac{p-1}{2}} x_2^1 x_3^{\frac{p-1}{2}} x_4^0$, is of degree p as desired. However, when $r = 0$ and $n = \frac{p+r+1}{2} = \frac{p+1}{2}$, then $r-2n = -(p+1)$ and $n-2r = \frac{p+1}{2}$. Now, replacing $-(p+1)$ by $p-1$, the corresponding term, $x_1^{\frac{p+1}{2}} x_2^{p-1} x_3^{\frac{p+1}{2}} x_4^0$, is of degree $2p$ and therefore to be discarded. We conclude then that when $r = 0$ an upper limit for n is $\frac{p+r}{2}$. The other limits may be checked in like manner.

In order to show that to each pair of values (r,n) there corresponds one and only one distinct term of the polynomial (except for $n = r = 0$), and at the same time check the corresponding intervals, consider the following reformulation - where as the integers k and L vary within their proper intervals, the (r,n) assume the same values as in (A).

For the case where $p = 6\alpha + 1$:

refer to (A1)

with r even

$$r = \frac{p-3k-1}{3};\ 0 \leqslant k \leqslant \frac{p-1}{3},\ \text{where } k \text{ is even}$$

$$n = \frac{p+r-2L-1}{2};\ 0 \leqslant L \leqslant \frac{3k}{2}$$

$$r-2n = -p+2I+1$$

$$n-2r = \frac{3k}{2} - L \quad \text{yielding}$$

$$F_1 = x_1^{\frac{2(p-1)}{3} - \frac{k}{2} - L}\ x_2^{2L+1}\ x_3^{\frac{3k}{2} - L}\ x_4^{\frac{p-1}{3} - k}$$

with r odd

$$r = \frac{p-3k-1}{3};\ 0 \leqslant k \leqslant \frac{p-1}{3},\ \text{where } k \text{ is odd}$$

$$n = \frac{p+r-2L}{2};\ 0 \leqslant L \leqslant \frac{3k+1}{2}$$

$$r-2n = -p+2L$$

$$n-2r = \frac{3k+1}{2} - L \quad \text{yielding}$$

$$F_2 = x_1^{\frac{4p-1}{6} - \frac{k}{2} - L}\ x_2^{2L}\ x_3^{\frac{3k+1}{2} - L}\ x_4^{\frac{p-1}{3} - k}$$

refer to (A2)

with r even

$$r = \frac{p-2k-1}{2};\ 0 \leqslant k \leqslant \frac{p-1}{2}$$

$$n = \frac{r-2L}{2};\ 0 \leqslant L \leqslant \frac{p-1}{4} - \frac{k}{2}$$

$$r - 2n = 2L$$

$$n-2r = \frac{p-1}{4} + \frac{3k+2}{2} - L - p \quad \text{yielding}$$

$$F_3 = x_1^{\frac{p-1}{4} - \frac{k}{2} - L}\ x_2^{2L}\ x_3^{\frac{p-1}{4} + \frac{3k+2}{2} - L}\ x_4^{\frac{p-1}{2} - k}$$

with r odd

$$r = \frac{p-2k-1}{2}; \ 0 \leq k \leq \frac{p-1}{2}$$

$$n = \frac{r-2L-1}{2}; \ 0 \leq L \leq \frac{p-3}{4} - \frac{k}{2}$$

$$r-2n = 2L + 1$$

$$n-2r = \frac{p-3}{4} + \frac{3k+2}{2} - L - p \quad \text{yielding}$$

$$F_4 = x_1^{\frac{p-3}{4} - \frac{k}{2} - L} x_2^{2L+1} x_3^{\frac{p-3}{4} + \frac{3k+2}{2} - L} x_4^{\frac{p-1}{2} - k}$$

refer to (A3)

with r even

$$r = \frac{2p-3k-2}{3}; \ 0 \leq k \leq \frac{p-7}{6}$$

$$n = \frac{r-2L}{2}; \ 0 \leq L \leq \frac{3k+2}{2}$$

$$r-2n = 2L$$

$$n-2r = \frac{3k+2}{2} - L - p \quad \text{yielding}$$

$$F_5 = x_1^{\frac{p-1}{3} - \frac{k}{2} - L} x_2^{2L} x_3^{\frac{3k+2}{2} - L} x_4^{\frac{2p-2}{3} - k}$$

with r odd

$$r = \frac{2p-3k-2}{3}; \ 0 \leq k \leq \frac{p-7}{6}$$

$$n = \frac{r-2L-1}{2}; \ 0 \leq L \leq \frac{3k+1}{2}$$

$$r-2n = 2L+1$$

$$n-2r = \frac{3k+1}{2} - L - p \quad \text{yielding}$$

$$F_6 = x_1^{\frac{2p-5}{6} - \frac{k}{2} - L} x_2^{2L+1} x_3^{\frac{3k+1}{2} - L} x_4^{\frac{2p-2}{3} - k}$$

For the case where $p = 6\alpha - 1$:

with r even, refer to (A1)

$$r = \frac{p-3k-2}{3};\ 0 \leq k \leq \frac{p-2}{3}$$

$$n = \frac{p+4-2L-1}{2};\ 0 \leq L \leq \frac{3k+1}{2}$$

$$r-2n = -p+2L+1$$

$$n-2r = \frac{3k+1}{2} - L \quad \text{yielding}$$

$$F_1' = x_1^{\frac{4p-5}{6} - \frac{k}{2} - L}\, x_2^{2L+1}\, x_3^{\frac{3k+1}{2} - L}\, x_4^{\frac{p-2}{3} - k}$$

with r odd

$$r = \frac{p-3k-2}{3};\ 0 \leq k \leq \frac{p-2}{3}$$

$$n = \frac{p+r-2L}{2};\ 0 \leq L \leq \frac{3k+2}{2}$$

$$r-2n = 2L-p$$

$$n-2r = \frac{3k+2}{2} - L \quad \text{yielding}$$

$$F_2' = x_1^{\frac{2p-1}{3} - \frac{k}{2} - L}\, x_2^{2L}\, x_3^{\frac{3k+2}{2} - L}\, x_4^{\frac{p-2}{3} - k}$$

refer to (A2)

r even

$$r = \frac{p-2k-1}{2};\ 0 \leq k \leq \frac{p-1}{2}$$

$$n = \frac{r-2L}{2}\ ;\ 0 \leq L \leq \frac{p-1}{4} - \frac{k}{2}$$

$$r-2n = 2L$$

$$n-2r = \frac{p-1}{4} + \frac{3k+2}{2} - L - p \quad \text{yielding}$$

$$F_3' = x_1^{\frac{p-1}{4} - \frac{k}{2} - L}\, x_2^{2L}\, x_3^{\frac{p-1}{4} + \frac{3k+2}{2} - L}\, x_4^{\frac{p-1}{2} - k}$$

with r odd

$$r = \frac{p-2k-1}{2};\ 0 \leqslant k \leqslant \frac{p-1}{2}$$

$$n = \frac{r-2L-1}{2};\ 0 \leqslant L \leqslant \frac{p-3}{4} - \frac{k}{2}$$

$$r-2n = 2L+1$$

$$n-2r = \frac{p-3}{4} + \frac{3k+2}{2} - L - p \quad \text{yielding}$$

$$F_4' = x_1^{\frac{p-3}{4} - \frac{k}{2} - L}\ x_2^{2L+1}\ x_3^{\frac{p-3}{4} + \frac{3k+2}{2} - L}\ x_4^{\frac{p-1}{2} - k}$$

refer to (A3)

with r even

$$r = \frac{2p-3k-1}{3};\ 0 \leqslant k \leqslant \frac{p-5}{6}$$

$$n = \frac{r-2L}{2};\ 0 \leqslant L \leqslant \frac{3k+1}{2}$$

$$r-2n = 2L$$

$$n-2r = \frac{3k+1}{3} - L - p \quad \text{yielding}$$

$$F_5' = x_1^{\frac{2p-1}{6} - \frac{k}{2} - L}\ x_2^{2L}\ x_3^{\frac{3k+1}{2} - L}\ x_4^{\frac{2p-1}{3} - k}$$

with r odd

$$r = \frac{2p-3k-1}{3};\ 0 \leqslant k \leqslant \frac{p-5}{6}$$

$$n = \frac{r-2L-1}{2};\ 0 \leqslant L \leqslant \frac{3k}{2}$$

$$r-2n = 2L+1$$

$$n-2r = \frac{3k}{2} - L - p \quad \text{yielding}$$

$$F_6' = x_1^{\frac{p-2}{3} - \frac{k}{2} - L}\ x_2^{2L+1}\ x_3^{\frac{3k}{2} - L}\ x_4^{\frac{2p-1}{3} - k}$$

In (A1) the minimum value for n is $2r$ while in (A2) n cannot exceed $\frac{r}{2}$. The case $n = r = 0$ has already been discussed.[4] For any other particular value of r, the x_1 exponents (which are denoted by n) in (A1) are distinct from the x_1 exponents in (A2). Hence the corresponding terms in (A1) are distinct from those of (A2). Similarly, r (the x_4 exponent) cannot exceed $\frac{p}{2}$ in (A2), and r cannot be less than $\frac{p+1}{2}$ in (A3). Hence the terms in (A2) are distinct from those in (A3). Finally, in (A1) the r cannot exceed $\frac{p}{3}$, and in (A3) the r cannot be less than $\frac{p+1}{2}$. Thus there are no terms in common.

Furthermore, (A1), (A2), or (A3) for a distinct value of k there corresponds a distinct value of the x_4 exponent and therefore a distinct term. Likewise, for each value of L there corresponds a distinct x_2 exponent. Then, as k and L vary, one obtains distinct pairs (n,r) where to each pair (n,r) there corresponds one and only one distinct term of the polynomial.

---

[4] If one uses the upper limits for k and L, the value $\frac{r}{2}$ is obtained. This, however, is the trivial case, $n = r \equiv 0$.

In order to check the invariants and arrive at a classification of the terms invariant under T, let us observe that where $p = 6\alpha + 1$

$$F_1 \sim_T E^{2L+1} E^{3k-2L} E^{p-1-3k} F_1 = E^p F_1 = F_1$$

$$F_2 \sim_T E^{2L} E^{3k+1-2L} E^{p-1-3k} F_2 = E^p F_2 = F_2$$

$$F_3 \sim_T E^{2L} E^{\frac{p-1}{2}+3k+2-2L} E^{\frac{3(p-1)}{2}-3k} F_3 = E^{2p} F_3 = F_3$$

$$F_4 \sim_T E^{2L+1} E^{\frac{p-3}{2}+3k+2-2L} E^{\frac{3(p-1)}{2}-3k} F_4 = E^{2p} F_4 = F_4$$

$$F_5 \sim_T E^{2L} E^{3k+2-2L} E^{2p-2-3k} F_5 = E^{2p} F_5 = F_5$$

$$F_6 \sim_T E^{2L+1} E^{3k+1-2L} E^{2p-2-3k} F_6 = E^{2p} F_6 = F_6$$

Similarly where $p = 6\alpha - 1$

$$F_1' \sim_T E^{2L+1} E^{3k+1-2L} E^{p-2-3k} F_1' = E^p F_1' = F_1'$$

$$F_2' \sim_T E^{2L} E^{3k+2-2L} E^{p-2-3k} F_2' = E^p F_2' = F_2'$$

$$F_3' \sim_T E^{2L} E^{\frac{p-1}{2}+3k+2-2L} E^{\frac{3(p-1)}{2}-3k} F_3' = E^{2p} F_3' = F_3'$$

$$F_4' \sim_T E^{2L+1} E^{\frac{p-3}{2}+3k+2-2L} E^{\frac{3(p-1)}{2}-3k} F_4' = E^{2p} F_4' = F_4'$$

$$F_5' \sim_T E^{2L} E^{3k+1-2L} E^{2p-1-3k} F_5' = E^{2p} F_5' = F_5'$$

$$F_6' \sim_T E^{2L+1} E^{3k-2L} E^{2p-1-3k} F_6' = E^{2p} F_6' = F_6'$$

By simple addition of the exponents it is seen that each F_i' or F_i $(i = 1, 2, 3, \ldots, 6)$ is of degree p. It remains only to find how many terms are under consideration.

To count the numbers of terms in (A1) for $p = 6\alpha + 1$, count the numbers of pairs (k, L) where

$$0 \leqslant k \leqslant \frac{p-1}{3}$$

$$0 \leqslant L \leqslant \frac{3k+1}{2}$$

$k = 0;$	$k = 1;$	$k = 2;$	$\ldots;\ k = \frac{p-1}{3}$
$0 \leqslant L \leqslant \frac{1}{2};$	$0 \leqslant L \leqslant 2;$	$0 \leqslant L \leqslant \frac{7}{2};$	$\ldots;\ 0 \leqslant L \leqslant \frac{p}{2}$
$L = 0;$	$L = 0, 1, 2;$	$L = 0$ to $3;$	$\ldots;\ L = 0$ to $\frac{p-1}{2}$
$S_n = 1\ +$	$3\ +$	$4\ +$	$\ldots + \frac{p+1}{2};\ n = \frac{p+2}{3}$

$$S_n = 1 + 3 + 4 + 6 + \ldots + \frac{p-1}{2} + \frac{p+1}{2}$$

$$S_n = \left(1 + 4 + 7 + \ldots + \frac{p+1}{2}\right) + \left(3 + 6 + 9 + \ldots + \frac{p-1}{2}\right)$$

$$S_n = \frac{p^2+6p+5}{12}$$

in (A2)

$$0 \leqslant k \leqslant \frac{p-1}{2}$$

$$0 \leqslant L \leqslant \frac{p-1}{2} - \frac{k}{2}$$

$k = \frac{p-1}{2};$	$k = \frac{p-3}{2};$	$k = \frac{p-5}{2};$	$\ldots;\ k = 0$
$0 \leqslant L \leqslant 0;$	$0 \leqslant L \leqslant \frac{1}{2};$	$0 \leqslant L \leqslant 1;$	$\ldots;\ 0 \leqslant L \leqslant \frac{p-1}{4}$
$L = 0;$	$L = 0;$	$L = 0, 1;$	$\ldots;\ L = 0$ to $\frac{p-1}{4}$ or $\frac{p-3}{4}$
$S_n = 1\ +$	$1\ +$	$2\ +$	$\ldots + \frac{p+3}{4}$ or $\frac{p+1}{4}$ and

$n = \frac{p+1}{2}$. If $\frac{p+1}{2} = \frac{6\alpha+2}{2} = 3\alpha+1$ is even, then α is odd and $S_n = 2\left(1 + 2 + 3 + \ldots + \frac{p+1}{4}\right) = \frac{p^2+6p+5}{16}$. If $\frac{p+1}{2}$ is odd, then α is even and

$$S_n = 2\left(1 + 2 + 3 + \ldots + \frac{p-1}{4}\right) + \frac{p+3}{4} = \frac{p^2+6p+9}{16}.$$

Similarly

in (A3) $\qquad 0 \leq k \leq \frac{p-7}{6}$

$$0 \leq L \leq \frac{3k+2}{2}$$

$k = 0; \quad k = 1; \quad k = 2; \quad \ldots; \; k = \frac{p-7}{6}$

$0 \leq L \leq 1; \; 0 \leq L \leq \frac{5}{2}; \quad 0 \leq L \leq 4; \quad \ldots; \; 0 \leq L \leq \frac{p-3}{4}$

$L = 0, 1; \; L = 0, 1, 2; \; L = 0 \text{ to } 4; \; \ldots; \; L = 0 \text{ to } \frac{p-3}{4} \text{ or } \frac{p-5}{4}$

$S_n = 2 + 3 + 5 + \ldots + \frac{p+1}{4}$ or $\frac{p-1}{4}$ where $n = \frac{p-1}{6}$.

For α odd, observe that S_n has an odd number of terms where the last term is $\frac{p+1}{4}$ since in (A2) the number of terms $\frac{p+1}{2}$ is even and hence $\frac{p+1}{4}$ is an integer.[5]

$$S_n = (2 + 5 + 8 + \ldots + \frac{p-11}{4} + \frac{p+1}{4}) + (3 + 6 + 9 + \ldots + \frac{p-7}{4})$$

$$S_n = \frac{p^2-10p+21}{96} + \frac{p^2-2p-35}{96} + \frac{p+1}{4}$$

$$S_n = \frac{p^2+6p+5}{48}.$$

For α even, S_n has an even number of terms and the last term is $\frac{p-1}{4}$ for in (A2) the last term is $\frac{p+3}{4}$ for α even therefore $\frac{p-1}{4}$ is an integer.

$$S_n = (2 + 5 + 8 + \ldots + \frac{p-5}{4}) + (3 + 6 + 9 + \ldots + \frac{p-1}{4})$$

$$S_n = \frac{p^2+2p-3}{96} + \frac{p^2+10p-11}{96} = \frac{p^2+6p-7}{48}.$$

In either case the number of terms in (A2) plus the number of terms in (A3) is $\frac{p^2+6p+5}{12}$. For $\frac{p^2+6p+5}{16} + \frac{p^2+6p+5}{48} = \frac{p^2+6p+5}{12}$ where α is odd and

[5] See Appendix B for examples.

$$\frac{p^2+6p+9}{16} + \frac{p^2+6p-7}{48} = \frac{p^2+6p+5}{12}$$ where α is even.

To count the number of terms for $p = 6\alpha - 1$ we again count the number of pairs (k,L)

in (A1) where $0 \leqslant k \leqslant \frac{p-2}{3}$ and $0 \leqslant L \leqslant \frac{3k+2}{2}$

$k = 0$; $k = 1$; $k = 2$; ...; $k = \frac{p-2}{3}$

$L = 0,1$; $L = 0, 1, 2$; $L = 0$ to 4; ...; $L = 0$ to $\frac{p-1}{2}$

$S_n = 2 + 3 + 5 + \ldots + \frac{p+1}{2}$

$n = \frac{p+1}{3}$ and $S_n = \frac{p^2+6p+5}{12}$

in (A2) where $0 \leqslant k \leqslant \frac{p-1}{2}$ and $0 \leqslant L \leqslant \frac{p-1}{4} - \frac{k}{2}$

$k = \frac{p-1}{2}$; $k = \frac{p-3}{2}$; $k = \frac{p-5}{2}$; $k = \frac{p-7}{2}$; ...; $k = 0$

$L = 0$; $L = 0$: $L = 0, 1$; $L = 0, 1$; ...; $L = 0$ to $\frac{p-3}{4}$ or $\frac{p-1}{4}$

$S_n = 1 + 1 + 2 + 2 + \ldots + \frac{p+3}{4}$ or $\frac{p+1}{4}$; $n = \frac{p+1}{2}$

in (A3) where $0 \leqslant k \leqslant \frac{p-5}{6}$ and $0 \leqslant L \leqslant \frac{3k+1}{2}$

$k = 0$; $k = 1$; $k = 2$; ...; $k = \frac{p-5}{6}$

$L = 0$: $L = 0$ to 2; $L = 0$ to 3; ...; $L = 0$ to $\frac{p-3}{4}$ or $\frac{p-5}{4}$

$S_n = 1 + 3 + 4 + \ldots + \frac{p+1}{4}$ or $\frac{p-1}{4}$

$n = \frac{p+1}{6}$. S_n in (A2) plus S_n in (A3) is equal to $\frac{p^2+6p+5}{12}$ as in the case where $p = 6\alpha + 1$.

The total number of terms thus obtained which are invariant under T is equal to

$$\frac{2(p^2+6p+5)}{12} + 2 = \frac{p^2+6p+17}{6} .$$

Of these terms there is one term containing E^0 as a factor under T. That term is x_1^p. There are $\frac{p^2+6p+5}{12}$ terms containing E^p, and likewise $\frac{p^2+6p+5}{12}$ terms containing E^{2p}. Finally the one term x_4^p contains E^{3p} under T. A simple concrete method for writing all of these terms for a given value of p has already been shown. (See Appendices A, C, and D for examples.)

CHAPTER III

THE MOST GENERAL POLYNOMIAL

A surface is still invariant if for each term $u_{abcd}\ x_1^a x_2^b x_3^c x_4^d$ of the polynomial used to represent it, it is true that

$$u_{abcd}\ x_1^a x_2^b x_3^c x_4^d \sim_T E^{s_{abcd}}\ u_{abcd} x_1^a x_2^b x_3^c x_4^d,$$

where each s_{abcd} is a member of the same congruence class $\bar{s} = s+(p)$. Therefore, let us now treat the most general such polynomial.

Again use is made of the general term $x_1^a x_2^b x_3^c x_4^d$ where a, b, c, and d are now representatives of the congruence classes $\bar{n} = n + (p)$, $\overline{r-2n-s} = r-2n-s+(p)$, $\overline{n-2r+s} = n-2r+s+(p)$ and $\bar{r} = r+(p)$ respectively. For any integer s in the interval $1 \leqslant s \leqslant p-1$:

(A'1) to $0 \leqslant r \leqslant \frac{s}{3}$ corresponds $p-s+2r \leqslant n \leqslant \frac{2p-s+r}{2}$

(A'2) to $0 \leqslant r \leqslant \frac{s}{2}$ corresponds $0 \leqslant n \leqslant \frac{p-s+r}{2}$

(A'3) to $\frac{s+1}{2} \leqslant r \leqslant \frac{p+s}{3}$ corresponds $2r-s \leqslant n \leqslant \frac{p-s+r}{2}$

(A'4) to $s \leqslant r \leqslant \frac{p+s}{2}$ corresponds $0 \leqslant n \leqslant \frac{r-s}{2}$

(A'5) to $\frac{p+s+1}{2} \leqslant r \leqslant \frac{2p+s}{3}$ corresponds $2r-s-p \leqslant n \leqslant \frac{r-s}{2}$

Terms of this form are of degree p or some multiple of p since

(1) $\bar{n} + \overline{r-2n-s} + \overline{n-2r+s} + \bar{r} \equiv \bar{0}$

Furthermore invariance is assured, since

(2) $x_1^a x_2^b x_3^c x_4^d \sim_T x_1^a E^b x_2^b E^{2c} x_3^c E^{3d} x_4^d$ and

(3) $\overline{r-2n-s} + 2\,\overline{n-2r+s} + 3\,\overline{r} = \overline{s}$

The limits in (A') are a simple extension of the limits in (A) and used in a similar manner.

TABLE 3. VALUES OF (n,r) FOR p=13 AND s=5

r = 0;	n = 0,1,2,3,4, 8,9,10
r = 1;	n = 0,1,2,3,4, 10,11
r = 2;	n = 0,1,2,3,4,5
r = 3;	n = 1,2,3,4,5
r = 4;	n = 3,4,5,6
r = 5;	n = 0, 5,6
r = 6;	n = 0, 7
r = 7;	n = 0,1
r = 8;	n = 0,1
r = 9;	n = 0,1,2
r =10;	n = 2

The derived polynomial has $\frac{p^2+6p+11}{6}$ terms. For, in this case, where p = 13, $\frac{13^2+6(13)+11}{6} = 43$ — the number of terms which may be derived (see Appendix C) from the above tabulation.

To count the number of terms in $(A'1)$, (that is to say in $0 \leqslant r \leqslant \frac{s}{3}$; corresponding to $p-s+2r \leqslant n \leqslant \frac{2p-s+r}{2}$) let $r = \frac{s-a_1-3k}{3}$; $0 \leqslant k \leqslant \frac{s-a_1}{3}$ where $s = a_1$ (mod. 3), $a_1 = 0, 1, 2$ and $n = \frac{2p-s+r-2L}{2}$; $0 \leqslant L \leqslant \frac{s-3r}{2} = \frac{3k+a_1}{2}$.

Consider the number of distinct pairs (k, L) where $s \equiv a_1$ (mod. 3), $a_1 = 0, 1, 2$ in

$$0 \leqslant k \leqslant \frac{s-a_1}{3};$$

$$0 \leqslant L \leqslant \frac{3k+a_1}{2}$$

When $s = 0$, then $a_1 = 0$, $k = 0$, $L = 0$, and we get one term of the polynomial.

Similarly,

$s = 1,\ a_1 = 1;$	$s = 2,\ a_1 = 2;$	$s = 3,\ a_1 = 0$	
$k = 0$	$k = 0$	$k = 0$	$k = 1$
$0 \leqslant L \leqslant \frac{1}{2}$	$0 \leqslant L \leqslant 1$	$0 \leqslant L \leqslant 0$	$0 \leqslant L \leqslant \frac{3}{2}$
$L = 0$	$L = 0, 1$	$L = 0$	$L = 0, 1$
1 term	2 terms	1 term	2 terms
		Total 3 terms	

$s = 4,\ a_1 = 1$		$s = 5,\ a_1 = 2$	
$k = 0,$	$k = 1$	$k = 0$	$k = 1$
$0 \leqslant L \leqslant \frac{1}{2}$	$0 \leqslant L \leqslant 2$	$0 \leqslant L \leqslant 1$	$0 \leqslant L \leqslant \frac{5}{2}$
$L = 0$	$L = 0, 1, 2$	$L = 0, 1$	$L = 0, 1, 2$
1 term	3 terms	2 terms	3 terms
Total 4 terms		Total 5 terms	

$s = 6,\ a_1 = 0$		
$k = 0$	$k = 1$	$k = 2$
$0 \leqslant L \leqslant 0$	$0 \leqslant 1 \leqslant \frac{3}{2}$	$0 \leqslant L \leqslant 3$
$L = 0$	$L = 0, 1$	$L = 0, 1, 2, 3$
1 term	2 terms	4 terms
	Total 7 terms	

$s = 7,\ a_1 = 1$		
$k = 0$	$k = 1$	$k = 2$
$0 \leqslant L \leqslant \frac{1}{2}$	$0 \leqslant L \leqslant 2$	$0 \leqslant L \leqslant \frac{7}{2}$
$L = 0$	$L = 0, 1, 2$	$L = 0, 1, 2, 3$
1 term	3 terms	4 terms
	Total 8 terms	

By studying the above cases and also those for

higher values of s, Table 4 is created, and the number of terms for each value of s is obtained, where $s < p$.

TABLE 4. THE NUMBER OF TERMS IN THE GENERAL POLYNOMIAL

s = 0;	1	= 1
s = 1;	1	= 1
s = 2;	2	= 2
s = 3;	1+2	= 3
s = 4;	1+3	= 4
s = 5;	2+3	= 5
s = 6;	1+2+4	= 6+1
s = 7;	1+3+4	= 7+1
s = 8;	2+3+5	= 8+2
s = 9;	1+2+4+5	= 9+3
s = 10;	1+3+4+6	= 10+4
s = 11;	2+3+5+6	= 11+5
s = 12;	1+2+4+5+7	= 12+6+1
s = 13;	1+3+4+6+7	= 13+7+1
s = 14;	2+3+5+6+8	= 14+8+2
s = 15;	1+2+4+5+7+8	= 15+9+3
s = 16;	1+3+4+6+7+9	= 16+10+4
s = 17;	2+3+5+6+8+9	= 17+11+5
s = 18;	1+2+4+5+7+8+10	= 18+12+6+1
s = 19;	1+3+4+6+7+9+10	= 19+13+7+1
s = 20;	2+3+5+6+8+9+11	= 20+14+8+2
s = 21;	1+2+4+5+7+8+10+11	= 21+15+9+3
s = 22;	1+3+4+6+7+9+10+12	= 22+16+10+4
s = 23;	2+3+5+6+8+9+11+12	= 23+17+11+5
s = 24;	1+2+4+5+7+8+10+11+13	= 24+18+12+6+1
s = 25;	1+3+4+6+7+9+10+12+13	= 25+19+13+7+1
s = 26;	2+3+5+6+8+9+11+12+14	= 26+20+14+8+2
s = 27;	1+2+4+5+7+8+10+11+13+14	= 27+21+15+9+3
s = 28;	1+3+4+6+7+9+10+12+13+15	= 28+22+16+10+4
s = 29;	2+3+5+6+8+9+11+12+14+15	= 29+23+17+11+5
s = 30;	1+2+4+5+7+8+10+11+13+14+16	= 30+24+18+12+6+1

In Table 4 the vertical pattern to the right suggests the use of six different forms for s. The identical pattern

to the left exhibits the series which are to be summed on the next pages.

In general the number of distinct pairs (k,L) in

$$0 \leqslant k \leqslant \frac{s-a_1}{3}\,; \quad \text{where } s = a_1 \pmod{3},$$

and

$$0 \leqslant L \leqslant \frac{3k+a_1}{2}\,; \qquad a_1 = 0,\ 1,\ 2$$

will be determined as follows.

For $s = 6B$, $a_1 = 0$, and $0 \leqslant k \leqslant 2B$

$k = 0$; $\quad k = 1$; $\quad k = 2$; $\quad k = 3$; $\quad \ldots$; $k = 2B$

$0 \leqslant L \leqslant 0$; $0 \leqslant L \leqslant \frac{3}{2}$; $0 \leqslant L \leqslant 3$; $0 \leqslant L \leqslant \frac{9}{2}$; $\ldots$; $0 \leqslant L \leqslant 3B$

$L = 0$; $\quad L = 0, 1$; $\quad L = 0$ to 3; $L = 0$ to 4; $\ldots$; $L = 0$ to $3B$

and one gets the sum.

$$S_n = 1 + 2 + 4 + 5 + 7 + \ldots + (3B-2) + (3B-1) + (3B+1);$$

where $n = 2B+1$. Thus,

$$S_n = (1 + 4 + 7 + \ldots + (3B-2) + (3B+1)) + (2 + 5 + 8 \ldots + (3B-1)) \text{ or}$$

$S_n = 3B^2 + 3B + 1$. In terms of s, the sum is

$$S_n = 3\left(\frac{s}{6}\right)^2 + 3\left(\frac{s}{6}\right) + 1 = \frac{s^2+6s+12}{12}$$

Now, for $s = 6B+1$, $a_1 = 1$, and $0 \leqslant k \leqslant 2B$

$k = 0$; $\quad k = 1$; $\quad k = 2$; $\quad \ldots$; $k = 2B$

$0 \leqslant L \leqslant \frac{1}{2}$; $0 \leqslant L \leqslant 2$; $0 \leqslant L \leqslant \frac{7}{2}$; $\ldots$; $0 \leqslant L \leqslant \frac{6B+1}{2}$

$L = 0$; $\quad L = 0$ to 2; $L = 0$ to 3; $\ldots$; $L = 0$ to $3B$

$$S_n = 1 + 3 + 4 + 6 + \ldots + 3B + (3B+1);\ n = 2B+1$$

$$S_n = (1 + 4 + 7 + \ldots + (3B-2) + (3B+1)) + (3 + 6 + 9 \ldots 3B)$$

$$S_n = 3B^2 + 4B + 1$$

$$S_n = 3\left(\frac{s-1}{6}\right)^2 + 4\left(\frac{s-1}{6}\right) + 1 = \frac{s^2+6s+5}{12}$$

For $s = 6B+2$, $a_1 = 2$, and $0 \leqslant k \leqslant 2B$

$k = 0$; $k = 1$; $k = 2$; ...; $k = 2B$

$0 \leqslant L \leqslant 1$; $0 \leqslant L \leqslant \frac{5}{2}$; $0 \leqslant L \leqslant 4$; ...; $0 \leqslant L \leqslant 3B+1$

$L = 0, 1$; $L = 0$ to 2; $L = 0$ to 4;...; $L = 0$ to $3B+1$

$S_n = 2 + 3 + 5 + 6 + \ldots + (3B) + (3B+2)$; $n = 2B+1$

$S_n = (2 + 5 + 8 + \ldots + (3B-1) + (3B+2)) + (3 + 6 + 9 \ldots 3B)$

$S_n = 3B^2 + 5B + 2$

$S_n = 3(\frac{s-2}{6})^2 + 5(\frac{s-2}{6}) + 2 = \frac{s^2+6s+8}{12}$.

For $s = 6B+3$, $a_1 = 0$, and $0 \leqslant k \leqslant 2B+1$

$k = 0$; $k = 1$; $k = 2$: ...; $k = 2B+1$

$0 \leqslant L \leqslant 0$; $0 \leqslant L \leqslant \frac{3}{2}$; $0 \leqslant L \leqslant 3$; ...; $0 \leqslant L \leqslant \frac{6B+3}{2}$

$L = 0$; $L = 0, 1$; $L = 0$ to 3; ...; $L = 0$ to $3B+1$

$S_n = 1 + 2 + 4 + 5 + \ldots + (3B+1) + (3B+2)$; $n = 2B + 2$

$S_n = (1 + 4 + 7 + \ldots + (3B+1)) + (2 + 5 + 8 + \ldots + (3B+2))$

$S_n = 3B^2 + 6B + 3$

$S_n = 3(\frac{s-3}{6})^2 + 6(\frac{s-3}{6}) + 3 = \frac{s^2+6s+9}{12}$.

For $s = 6B+4$, $a_1 = 1$, and $0 \leqslant k \leqslant 2B+1$

$k = 0$; $k = 1$; $k = 2$; ...; $k = 2B+1$

$0 \leqslant L \leqslant \frac{1}{2}$; $0 \leqslant L \leqslant 2$; $0 \leqslant L \leqslant \frac{7}{2}$; ...; $0 \leqslant L \leqslant 3B+2$

$L = 0$; $L = 0,1,2$; $L = 0$ to 3; ...; $L = 0$ to $3B+2$

$S_n = 1 + 3 + 4 + 6 + \ldots + (3B+1) + (3B+3)$; $n = 2B+2$

$S_n = (1 + 4 + 7 + \ldots + (3B+1)) + (3 + 6 + 9 + \ldots + (3B+3))$

$S_n = 3B^2 + 7B + 4$

$S_n = 3(\frac{s-4}{6})^2 + 7(\frac{s-4}{6}) + 4 = \frac{s^2+6s+8}{12}$.

For $s = 6B+5$, $a_1 = 2$, and $0 \leq k \leq 2B+1$

$k = 0$; $k = 1$; $k = 2$; ...; $k = 2B+1$

$0 \leq L \leq 1$; $0 \leq L \leq \frac{5}{2}$; $0 \leq L \leq 4$; ...; $0 \leq L \leq \frac{6B+5}{2}$

$L = 0, 1$; $L = 0$ to 2; $L = 0$ to 4; ...; $L = 0$ to $3B+2$

$$S_n = 2 + 3 + 5 + 6 + \ldots + (3B+2) + (3B+3); \ n = 2B+2$$

$$S_n = (2 + 5 + 8 + \ldots + (3B+2)) + (3 + 6 + 9 + \ldots + (3B+3))$$

$$S_n = 3B^2 + 8B + 5$$

$$S_n = 3\left(\frac{s-5}{6}\right)^2 + 8\left(\frac{s-5}{6}\right) + 5 = \frac{s^2+6s+5}{12}$$

By summing finite arithmetic series in the manner just demonstrated, we find that when $0 \leq s' \leq p$, there are $H(s) = \dfrac{s^2+6s+h_i}{12}$ terms corresponding to $(A'1)$. Similarly, when $p \leq s' \leq 2p$, there are $G(s) = \dfrac{p^2+(6+2s)p-2s^2+g_i}{12}$ terms from $(A'2)$ and $(A'3)$. Finally, when $2p \leq s' \leq 3p$, there are $Q(s)$ terms where $Q(s) = \dfrac{p^2+(6-2s)p+s^2-6s+q_i}{12}$, taken from $(A'4)$ and $(A'5)$. It is given that $s' \equiv s(\text{mod. } p)$, $s = 6B+i$, $i = 0, 1, 2, 3, 4, 5$, and s lies in the interval $1 \leq s \leq p-1$. For $p = 6\alpha-1$ and $s' = s$, then h_0 to h_5 equal 12, 5, 8, 9, 8 and 5, respectively. Similarly, if $s' = s+p$, then g_0 to g_5 equal 5, 9, 5, 5, 9, and 5, and if $s' = s+2p$, then q_0 to q_5 equal 5, 8, 9, 8, 5, and 12.

For $p = 6\alpha+1$ and $s' = s$, then h_0 to h_5 equal 12, 5, 8, 9, 8 and 5, respectively. Similarly, if $s' = s+p$, then g_0 to h_5 equal 5, 5, 9, 5, 5, and 9, and if $s' = s+2p$, then q_0 to q_5 equal 5, 12, 5, 8, 9, and 8.

As in the previous discussion these terms are all distinct. There are then, $H(s) + G(s) + Q(s) = \frac{p^2+6p+11}{6}$ terms in the most general polynomial for each invariant surface corresponding to each value of s in the interval $1 \leqslant s \leqslant (p-1)$.

There were $\frac{p^2+6p+17}{6}$ terms in the invariant polynomial[1] which was discussed in some detail earlier. Our method now includes all the terms of the general homogeneous polynomial, for by simple addition $\frac{p^2+6p+17}{6} + \frac{(p-1)(p^2+6p+11)}{6} = \frac{(p+1)(p+2)(p+3)}{6}$. This is the total number of terms in the most general polynomial.[2]

The method described in this chapter may be used for writing the terms of any homogeneous polynomial of prime degree in four variables.[3] At the same time it permits a classification of the terms of the polynomial

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[1]If s assumes the value 0 in the above general formulation, one obtains this invariant polynomial.

[2]For derivation of this formula see Snyder, V. and Sisam, C. H., Analytic Geometry of Space, Henry Holt, New York, 1914.

[3]Refer to Appendix A, C, and D.

into sets of $\frac{p^2+6p+11}{6}$ terms each (except for the first set which has one extra term) where to each set there corresponds an invariant surface. This direct approach constructively accounts for all terms and thus also affords a new way to count the number of terms in the most general polynomial of prime degree.

CHAPTER IV

PERFECT POINTS OF INVOLUTION WITHIN NEIGHBORHOODS OF ORDER TWENTY OR BELOW

1. An Algebraic Surface.

The locus of the equation

$$(1) \qquad f(x) = \sum \frac{n!}{\alpha!\,\beta!\,\gamma!\,\delta!}\, a_{\alpha\beta\gamma\delta}\, x_1^{\alpha} x_2^{\beta} x_3^{\gamma} x_4^{\delta} = 0$$

wherein α, β, γ, δ are positive integers (or zero) satisfying the equation $\alpha + \beta + \gamma + \delta = n$, is called an algebraic surface[1] of degree n. In Chapter II the terms of these polynomials with $n = p$ have been classified so that one can now readily and with absolute assurance select a surface for study which is invariant under the birational transformation, T

$$(1') \qquad x_1':x_2':x_3':x_4' = x_1:Ex_2:E^2x_3:E^3x_4 \quad \text{where } E^p = 1.$$

Such a surface F is said to contain the involution I_p of order p in that it contains the p points (x_1, x_2, x_3, x_4), $(x_1, E^1x_2, E^2x_3, E^3x_4)$, $(x_1, E^2x_2, E^4x_3, E^6x_4)$, ... , $(x_1, E^ix_2, E^{2i}x_3, E^{3i}x_4)$, ... , $(x_1, E^{p-1}x_2, E^{2p-2}x_3, E^{3p-3}x_4)$ in any one of the

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[1]This definition is taken from Snyder and Sisam, Analytic Geometry of Space, p. 206.

groups of the involution.

2. Fundamental Points.

The points $P_1(1, 0, 0, 0)$, $P_2(0, 1, 0, 0)$, $P_3(0, 0, 1, 0)$, and $P_4(0, 0, 0, 1)$ are called points of coincidence since $P_i \sim_T P_i$, $i = 1, 2, 3, 4$. Any one of these points P is called a fundamental point of the birational transformation Q if $P \sim_Q (0, 0, 0, 0)$ and it is said to have no transform.

3. Perfect Points.

One says that an invariant point p of I_p is a point of perfect coincidence if to a curve traced on F passing through this point, the homography T makes to correspond a curve with the same tangent as the first at that point.[2] The point is non perfect in the opposite case.

One is interested in perfect points by the very nature of the study, as well as in the contributions to be made in studying the nature of singularity. For example, if a point is a point of perfect coincidence, it is not only coincident but each direction of a curve emanating from the point is invariant. Such a point is

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[2]This is a translation from Godeaux, L., "Sur les Homographies Planes Cycliques," Memoires de La Societe des Sciences de Liege, t. XV, 1930, pp. 1, 2.

described as having invariant points in the first order neighborhoods in any direction.

4. First Order Neighborhoods.

A point P_t on F' is said to be the homologue of a point in the first order neighborhood of the point P, (a fundamental point of the birational transformation of F into F') if to a point X in the direction t from P, there corresponds by the transformation, a point Q such that as X approaches P as a limit in the direction t, Q approaches P_t as a limit. Points in the second, third, and higher order neighborhoods can be defined by successive birational transformations of F into F'.

Two points are always invariant under I_p', the involution of order p transforming F' into itself, which corresponds to I_p using the transformation T. These two points correspond to the two invariant directions at P.

For example:

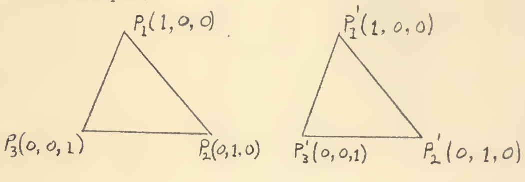

Figure 1

Given the two triangles, $P_1P_2P_3$ and $P_1'P_2'P_3'$. Assume one has a quadratic transformation U which causes P_1 to go into (0,0,0), but U^{-1} causes $P_1'(1,0,0)$ to go into $P_1(1,0,0)$. If one takes the limiting position of any point Q on the line P_1P_3 as it approaches point P_1, and one finds that this limiting position of point Q goes into, by use of U, the point $P_1'(1,0,0)$ then the point P_1' is the homologue of the point in the first order neighborhood of P_1 along the invariant direction P_1P_3.

Godeaux[3] defines a first order neighborhood as stated in this paragraph. Consider a curve C having an ordinary point at M and the tangent m to the curve at this point. In general, the curve C and the line m have two points indistinguishably thrown together in the intersection at M. We will agree to express this property by saying that the curve C and the line m have in common the point M and a fictitious point infinitely near M. When the curve C varies, always passing simply through M, the tangent m describes the pencil of lines with vertex M and one obtains a cluster of fictitious points infinitely near M which one calls the first order neighborhood of the point M.

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[3]Godeaux, Lucien, Geometrie Algebrique, Liege Sciences et Lettres, 1948, tome 1, p. 24 (translated).

The study of united (invariant) points is quite general since by a projective transformation any point may be chosen as a united point with respect to a new coordinate system.

5. The Surface F_4.

Now if a united point of a surface F is not perfect, the purpose of this chapter is to discover in which order of neighborhood of the non perfect point does one find perfect points. For this study consider the surface[4] (2) $F_4(x_1,x_2,x_3,x_4) = ax_2x_3^3 + bx_1x_2x_4^2 + cx_1x_3^2x_4 + dx_2^2x_3x_4 = 0$. It is invariant under the cyclic collineation T of prime period p.

(1') $T;\ x_1':x_2':x_3':x_4' = x_1:Ex_2:E^2x_3:E^3x_4$ where $E^p = 1$

since $F_4 \sim_T E^7F_4$.

The surface F_4 then contains involutions of all periods. Therefore a study of points on this surface can be conducted in a unique manner, and should be made in some detail.

Points $P_1(1,0,0,0)$, $P_2(0,1,0,0)$, $P_3(0,0,1,0)$ and $P_4(0,0,0,1)$ are all invariant under T and lie on the surface F_4. This fact may be stated in the following

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[4] Hutcherson, W. R., "A Cyclic Involution of Period Eleven," Canadian Journal of Mathematics, Vol. III, No. 2, 1951.

theorem.

THEOREM 1. <u>Each vertex of the tetrahedron of reference not only lies on the surface but is a point of coincidence</u>.

By rewriting F_4 in the order

$$ax_2x_3^3 + x_4(bx_1x_2x_4 + cx_1x_3^2 + dx_2^2x_3) = 0$$

it is easily seen that the line $P_1P_2(x_3 = x_4 = 0)$ lies on the surface. However, only the two points P_1 and P_2 of the line are invariant under T. In similar manner P_1P_4, P_1P_3, P_2P_4, and P_3P_4 lie on F_4 with only two invariant points on each line. The line P_2P_3 does not lie on the surface. A second theorem has been proved.

THEOREM 2. <u>This surface includes all the six edges of the tetrahedron of reference, except</u> P_2P_3.

It is true that P_3 is simple on F_4 (having a single tangent plane, $x_2 = 0$) at P_3, while P_2 and P_4 are double (having two tangent planes). The two tangent planes at P_2 are $x_3 = 0$, and $x_4 = 0$, and at P_4 they are $x_1 = 0$ and $x_2 = 0$. The point P_1 is triple, the tangent planes being $x_4 = 0$, $x_3 = \pm\sqrt{-\frac{b}{c}x_2x_4}$. The point P_3 shall be examined in detail.

Consider a curve C, not transformed into itself by T, and passing through P_3. Notice that the line P_2P_3 does not lie on F_4, neither does it act as a tangent

to the curve C for if it did one would have an invariant tangent under T and should pick some other curve C.

Take the plane $x_4 + K_1x_1 = 0$, of the pencil passing through P_2 and P_3, which is tangent to C (or contains the line tangent to C at P_3). This plane is transformed into $E^3x_4 + K_1x_1 = 0$ or $x_4 + K_1E^{p-3}x_1 = 0$ by T and hence is non-invariant. Since the plane containing the tangent to C at P_3 is non-invariant under T, then neither is the tangent invariant. Now since C was a variable curve through P_3 satisfying the non-invariant property, it follows that P_3 is an imperfect point of coincidence. In a similar manner it may be shown that P_1, P_2, and P_4 are also imperfect coincidence points. The following theorem has been proved.

THEOREM 3. The I_p belonging to F_4 in S_3 (three space) has four imperfect points of coincidence.

6. The Complete System of Curves $|A|$.

Consider the complete system of curves $|A|$ cut out on F_4 by all surfaces of order p (where p is a fixed prime).

To discover the restrictions placed upon u and v (positive integers where $v \geqq u$) in order that the dimension of the complete system $|A|$ cut out on F_u by all surfaces of prime order v, shall be equal to its genus, let

$u = 4$. Then

(3) $6u^2v - 24uv = 0 = u^3 - 6u^2 + 11u - 12$

Adding $6 + 3uv^2 - 3uv^2 + 12uv$ to both sides of the equation and rearranging the terms one obtains

(4) $-9uv + 6 + 3uv^2 + 3u^2v - 3uv = u^3 + 3uv^2 - 3vu^2$
$+12uv - 6u^2 + 11u - 6.$

Adding $3u^2v^2$ and subtracting the same quantity, $3u^2v^2$, on the left side of (4), and likewise adding and subtracting $v^3 + 6v^2 + 11v + 6$ on the right side one sees that

(5) $3\left[u^2v^2 - 3uv + 2\right] - 3\left[uv(uv-v-u+1)\right] =$
$v^3+6v^2+11v+6 - \left[(v-u)^3 + 6(v-u)^2 + 11(v-u)\right] - 12.$

Now divide both sides by 6 and notice that upon factoring

(6) $\dfrac{(uv-1)(uv-2)}{2} - \dfrac{uv(u-1)(v-1)}{2} =$
$\dfrac{(v+1)(v+2)(v+3)}{6} - \dfrac{(v-u+1)(v-u+2)(v-u+3)}{6} - 1$

When $u = 4$, the dimension of $|A|$ is equal to its genus, since the expression for dimension on the right side of (6) is balanced on the left by the corresponding expression for genus.[5]

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[5] Godeaux, Lucien. Geometrie Algebrique, 1948, tome 1, p. 38. (In the cases considered here the number of nodes is zero, causing a reduction in the proper formula concerning the genus of a space curve.)

The steps on the preceding page are all retraceable leading back to equation (3). Assuming that the dimension is equal to the genus, then

$$(3) \quad 6u^2v - 24uv = u^3 - 6u^2 + 11u - 12.$$

Now, assuming also that $u \neq 4$ and solving for v, one finds

$$(7) \quad v = \frac{u^3-6u^2+11u-12}{6u^2-24u} = \frac{(u-4)(u-2u+3)}{(u-4)(6u)} \quad \text{or}$$

$$(8) \quad v = \frac{u^2-2u+3}{6u}$$

But we have supposed that $v \geqslant u$ where u and v are positive integers. This requires that

$$(9) \quad \frac{u^2-2u+3}{6u} \geqslant u$$

or

$$(10) \quad u^2 - 2u + 3 \geqslant 6u^2$$

This inequality obviously has no solutions in u. Therefore, we have proved the following:

THEOREM 4. *A necessary and sufficient condition that the complete system of curves* $|A|$ *(cut out on* F_u *by all surfaces of order* v*) shall have its genus equal to its dimension is that* $u = 4$.

The complete system of curves $|A|$ cut out on F_4

is of dimension

$$\frac{(p+1)(p+2)(p+3)}{6} - \frac{(p-4+1)(p-4+2)(p-4+3)}{6} - 1 \quad \text{or}$$

$$\frac{(p+1)(p+2)(p+3)}{6} - \frac{(p-1)(p-2)(p-3)}{6} - 1 = 2p^2 + 1.$$

Then the dimension of $|A|$ is equal to its genus and both are equal to $2p^2+1$.

A curve A of this system is not in general transformed into itself by T. There are, however, p partial systems $|A_i|$ in $|A|$ which are transformed into themselves. See Chapter III for the p partial systems each corresponding to a value of s in the interval, $0 \leq s \leq p-1$.

Referring the curves A_1 projectively[6] to the hyperplanes of a linear space of $\frac{p^2+6p+17}{6} - 1 = \frac{p^2+6p+11}{6}$ dimensions, one obtains a surface φ_1 of order 4p, as the image of I_p. If we refer the curves A_i projectively to the hyperplanes of a linear space of $\frac{p^2+6p+11}{6} - 1 = \frac{p^2+6p+5}{6}$ dimensions, we obtain a surface φ_i again of order 4p, as the image of I_p for each value of $i = 2, 3, 4, \ldots, p$.

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[6]Hutcherson, W. R., _op. cit._, p. 156.

A detailed study of one of these image surfaces φ_i, and the points on φ_i corresponding to the coincident points on F, intimately involves the study of perfect points.[7] Though this work does not delve into these properties, it furthers the ground work by a treatment of perfect points on F_4 at ρ_3 which is rather general and capable of extension to other F surfaces and other coincident points.

7. <u>Use of Quadratic Transformations</u>.

In the tangent plane $x_2 = 0$, the involution I_p is generated by the homography T, which is

(11) $x_1':x_3':x_4' = x_1:E^2x_3:E^3x_4$.

By the successive use of the quadratic transformation, A,

(12) $z_1:z_3:z_4 = x_4^2:x_3x_4:x_1x_3$ and A^{-1}

(13) $x_1:x_3:x_4 = z_1z_4:z_3^2:z_1z_3$ one gets

(14) $T_1 = ATA^{-1}$, $T_2 = AT_1A^{-1}$, $T_3 = AT_2A^{-1}, \ldots, T_i = AT_{i-1}A^{-1}$

The number of successive uses of the transformation, A, in this manner determines the order of the neighborhood at ρ_3 in which a perfect point is found. The notation for a point in the first order neighborhood is ρ_{34}, if ρ_3 is

[7] Godeaux, L., "Sur les Points de Diramation Isoles des Surfaces Multiples," <u>Bulletin de l'Academie Royale de Belgique (Classe de Sciences)</u> 5^e Serie 35:636-641, 828-833, 1949.

approached along the line $O_3O_4(x_1 = x_2 = 0)$.

Since our surface F_4 contains involutions of all orders, it is possible to consider a number of cases at the same time. For example,

(15) $(x_1,x_3,x_4) \sim_A (x_4^2,x_3x_4,x_1x_3) \sim_T$

$(E^6x_4^2,E^5x_3x_4,E^2x_1x_3) \sim_{A^{-1}} (E^6x_1,E^5x_3,E^2x_4)$,

which becomes

(16) $T_1;\ x_1':x_3':x_4' = E^6x_1:E^5x_3:E^2x_4$ or

(17) $T_1;\ x_1':x_3':x_4' = E^4x_1:E^3x_3:x_4$

This transformation, T_1, gives evidence of an imperfect point at ρ_{34} for all primes except when $p = 2$. In this case the tangent plane $x_4 + k_1x_1 = 0$ is transformed into $x_4+E^4k_1x_1 = 0$ or $x_4 + k_1x_1 = 0$ since $E^2 = 1$. Thus, this plane containing the tangent line is invariant when $p = 2$, but not invariant when p is any other prime. Since the other plane, $(x_2 = 0)$ which also contains the tangent line, is a fixed plane of reference, it follows that the line of intersection of these two planes (the tangent line to the arbitrary curve C at P_{34}) is either invariant or not according as $p = 2$ or not. Hence, P_{34} is a perfect point with respect to the involution I_2 but imperfect for any other involution I_p.

Repeating the process

(18) $(x_1,x_3,x_4) \sim_A (x_4^2, x_3x_4, x_1x_3) \sim_{T_1} (E^4x_4^2, E^7x_3x_4, E^{11}x_1x_3)$

and $(E^4x_4^2, E^7x_3x_4, E^{11}x_1x_3) \sim_{A^{-1}} (E^4x_1, E^7x_3, E^{11}x_4)$ or

(19) T_2; $x_1':x_3':x_4' = E^4x_1, E^7x_3, E^{11}x_4$ which reduces to

(20) T_2; $x_1':x_3':x_4' = E^0x_1, E^3x_3, E^7x_4$ and

(21) $x_4 + K_1x_1 = 0 \sim_{T_2} E^7x_4 + K_1x_1 = 0.$

The transformation, T_2, gives evidence of an imperfect point in the second order neighborhood of P_3 at P_{344} for all primes except $p = 7$ where P_{344} is perfect.

Calculate T_3 in the same manner,

(22) $(x_1,x_3,x_4) \sim_A (x_4^2, x_3x_4, x_1x_3) \sim_{T_2} (E^{22}x_4^2, E^{18}x_3x_4, E^{11}x_1x_3)$

$(E^{22}x_4^2, E^{18}x_3x_4, E^{11}x_1x_3) \sim_{A^{-1}} (E^{22}x_1, E^{18}x_3, E^{11}x_4)$

(23) T_3; $x_1':x_3':x_4' = E^{22}x_1 : E^{18}x_3 : E^{11}x_4$ or

(24) T_3; $x_1':x_3':x_4' = E^{11}x_1 : E^7x_3 : E^0x_4$ and

(25) $x_4 + K_1x_1 = 0 \sim_{T_3} x_4 + E^{11}K_1x_1 = 0$

The transformation, T_3, gives evidence of an imperfect point in the third order neighborhood of P_3 at P_{3444} for all primes except for $p = 11$ (under all involutions except I_{11}) where P_{3444} is perfect.

8. Perfect Points Found by I.B.M. Calculator.

In order to make this study as inclusive as possible this quadratic reduction process was set up for the International Business Machine calculator. In Table 5 the numbers in the first three columns represent the exponents on E for the coefficients of x_1, x_3, and x_4 respectively, in the successive transformations. The fourth column contains the difference d for which $E^d = 1$ is the criterion for a perfect point.

The successive transformations, $T_i = AT_{i-1}A^{-1}$, are determined where the i^{th} row of numbers corresponds to the transformation T_i in Table 5.

TABLE 5. I.B.M. CALCULATION

0000000000000	0000000000002	0000000000003	0000000000003
0000000000006	0000000000005	0000000000002	0000000000004
0000000000004	0000000000007	0000000000011	0000000000007
0000000000022	0000000000018	0000000000011	0000000000011
0000000000022	0000000000029	0000000000040	0000000000018
0000000000080	0000000000069	0000000000051	0000000000029
0000000000102	0000000000120	0000000000149	0000000000047
0000000000298	0000000000269	0000000000222	0000000000076
0000000000444	0000000000491	0000000000567	0000000000123
0000000001134	0000000001058	0000000000935	0000000000199
0000000001870	0000000001993	0000000002192	0000000000322
0000000004384	0000000004185	0000000003863	0000000000521
0000000007726	0000000008048	0000000008569	0000000000843
0000000017138	0000000016617	0000000015774	0000000001364
0000000031548	0000000032391	0000000033755	0000000002207
0000000067510	0000000066146	0000000063939	0000000003571
0000000127878	0000000130085	0000000133656	0000000005778
0000000267312	0000000263741	0000000257963	0000000009349
0000000515926	0000000521704	0000000531053	0000000015127
0000001062106	0000001052757	0000001037630	0000000024476
0000002075260	0000002090387	0000002114863	0000000039603

For example:

T : 0, 2, 3; 3 where if $E^3 = 1$, then ρ_3 is perfect.

T_1: 6, 5, 2; 4 where if $E^4 = 1$, then ρ_{34} is perfect.

T_2: 4, 7,11; 7 where if $E^7 = 1$, then ρ_{344} is perfect.

T_3:22,18,11; 11 where if $E^{11} = 1$, then ρ_{3444} is perfect.

Observing this I.B.M. data pertaining to the first twenty neighborhoods, one derives Table 6.

TABLE 6. BRIEF SUMMARY OF FIRST TWENTY NEIGHBORHOODS[8] ALONG THE P_3P_4 DIRECTION

P^{3}_{3}			$P^{521}_{34_{11}}$		
$P^{2}_{34_{1}}$			$P^{3}_{34_{12}}$	$P^{281}_{34_{12}}$	
$P^{7}_{34_{2}}$			$P^{2}_{34_{13}}$	$P^{11}_{34_{13}}$	$P^{31}_{34_{13}}$
$P^{11}_{34_{3}}$			$P^{2207}_{34_{14}}$		
$P^{2}_{34_{4}}$	$P^{3}_{34_{4}}$		$P^{3571}_{34_{15}}$		
$P^{29}_{34_{5}}$			$P^{2}_{34_{16}}$	$P^{3}_{34_{16}}$	$P^{107}_{34_{16}}$
$P^{47}_{34_{6}}$			$P^{9349}_{34_{17}}$		
$P^{2}_{34_{7}}$	$P^{19}_{34_{7}}$		$P^{7}_{34_{18}}$	$P^{2161}_{34_{18}}$	
$P^{3}_{34_{8}}$	$P^{41}_{34_{8}}$		$P^{2}_{34_{19}}$	$P^{29}_{34_{19}}$	$P^{211}_{34_{19}}$
$P^{199}_{34_{9}}$			$P^{3}_{34_{20}}$	$P^{43}_{34_{20}}$	$P^{307}_{34_{20}}$
$P^{2}_{34_{10}}$	$P^{7}_{34_{10}}$	$P^{23}_{34_{10}}$			

[8] $P^{p}_{34_{i}}$ means P_3 has a perfect point, under I_p, in the i^{th} order neighborhood along the direction $x_1=x_2=0$.

By the successive use of the transformation B,

(26) $\quad z_1:z_3:z_4 = x_3x_4:x_1x_3:x_1^2,$

and B^{-1},

(27) $\quad x_1:x_3:x_4 = z_3z_4:z_3^2:z_1z_4,$

one gets the $T_j = BT_{j-1}B^{-1}$ transformation which determines the criterion to be used to obtain perfect points in the j^{th} order neighborhood of P_3 along the direction $x_2=x_4=0$. The results of this study can be stated concisely in Table 7.

TABLE 7. BRIEF SUMMARY OF FIRST TWENTY NEIGHBORHOODS[9] ALONG THE P_3P_1 DIRECTION

P^{3}_{3}					
$P^{5}_{31_1}$			$P^{937}_{31_{11}}$		
$P^{2}_{31_2}$			$P^{37}_{31_{12}}$	$P^{41}_{31_{12}}$	
$P^{13}_{31_3}$			$P^{2}_{31_{13}}$	$P^{3}_{31_{13}}$	$P^{409}_{31_{13}}$
$P^{3}_{31_4}$	$P^{7}_{31_4}$		$P^{11}_{31_{14}}$	$P^{19}_{31_{14}}$	
$P^{2}_{31_5}$	$P^{17}_{31_5}$		$P^{3}_{31_{15}}$	$P^{5}_{31_{15}}$	$P^{29}_{31_{15}}$
$P^{5}_{31_6}$	$P^{11}_{31_6}$		$P^{2}_{31_{16}}$	$P^{23}_{31_{16}}$	$P^{113}_{31_{16}}$
$P^{2}_{31_7}$	$P^{3}_{31_7}$		$P^{3}_{31_{17}}$	$P^{7}_{31_{17}}$	$P^{89}_{31_{17}}$
$P^{233}_{31_8}$			$P^{17}_{31_{18}}$	$P^{1601}_{31_{18}}$	
$P^{13}_{31_9}$	$P^{29}_{31_9}$		$P^{2}_{31_{19}}$	$P^{97}_{31_{19}}$	$P^{227}_{31_{19}}$
$P^{2}_{31_{10}}$	$P^{5}_{31_{10}}$	$P^{61}_{31_{10}}$	$P^{5}_{31_{20}}$	$P^{14251}_{31_{20}}$	

9 $P^{p}_{31_j}$ means P_3 has a perfect point, under I_p, in the j^{th} order neighborhood along the direction $x_2 = x_4 = 0$.

When the quadratic transformations A and A^{-1} are used alternately with the transformations B and B^{-1}, Table 8 of perfect points results.

TABLE 8. BRIEF SUMMARY OF THE FIRST TWENTY NEIGHBORHOODS USING ALTERNATE REDUCTION PROCESSES

P^{3}_{3}			
$P^{5}_{31_{1}}$		$P^{5}_{3141\ldots_{11}}$	
$P^{7}_{314_{2}}$		$P^{3}_{3141\ldots_{12}}$	
$P^{3}_{3141_{3}}$		$P^{29}_{3141\ldots_{13}}$	
$P^{11}_{31414_{4}}$		$P^{31}_{3141\ldots_{14}}$	
$P^{13}_{314141_{5}}$		$P^{3}_{3141\ldots_{15}}$	$P^{11}_{3141\ldots_{15}}$
$P^{3}_{3141414_{6}}$	$P^{5}_{3141\ldots_{6}}$	$P^{5}_{3141\ldots_{16}}$	$P^{7}_{3141\ldots_{16}}$
$P^{17}_{3141\ldots_{7}}$		$P^{37}_{3141\ldots_{17}}$	
$P^{19}_{3141\ldots_{8}}$		$P^{3}_{3141\ldots_{18}}$	$P^{13}_{3141\ldots_{18}}$
$P^{3}_{3141\ldots_{9}}$	$P^{7}_{3141\ldots_{9}}$	$P^{41}_{3141\ldots_{19}}$	
$P^{23}_{3141\ldots_{10}}$		$P^{43}_{3141\ldots_{20}}$	

When the quadratic transformations B and B^{-1} are used alternately with the transformations A and A^{-1}, Table 9 of perfect points results.

TABLE 9. BRIEF SUMMARY OF THE FIRST TWENTY NEIGHBORHOODS USING ALTERNATE REDUCTION PROCESSES

P^{3}_{3}			
$P^{2}_{34_1}$		$P^{2}_{3414\ldots_{11}}$	$P^{7}_{3414\ldots_{11}}$
$P^{5}_{341_2}$		$P^{3}_{3414\ldots_{12}}$	$P^{5}_{3414\ldots_{12}}$
$P^{2}_{3414_3}$	$P^{3}_{3414\ldots_{3}}$	$P^{2}_{3414\ldots_{13}}$	
$P^{7}_{34141_4}$		$P^{17}_{3414\ldots_{14}}$	
$P^{2}_{341414_5}$		$P^{2}_{3414\ldots_{15}}$	$P^{3}_{3414\ldots_{15}}$
$P^{3}_{3414141_6}$		$P^{19}_{3414\ldots_{16}}$	
$P^{2}_{3414\ldots_{7}}$	$P^{5}_{3414\ldots_{7}}$	$P^{2}_{3414\ldots_{17}}$	$P^{5}_{3414\ldots_{17}}$
$P^{11}_{3414\ldots_{8}}$		$P^{3}_{3414\ldots_{18}}$	$P^{7}_{3414\ldots_{18}}$
$P^{2}_{3414\ldots_{9}}$	$P^{3}_{3414\ldots_{9}}$	$P^{2}_{3414\ldots_{19}}$	$P^{11}_{3414\ldots_{19}}$
$P^{13}_{3414\ldots_{10}}$		$P^{23}_{3414\ldots_{20}}$	

9. Complete Tabulation of Perfect Points

For the complete picture of all of the possible combinations of the quadratic reduction processes related to the transformations A and B, the results are first given for neighborhoods, one, two, three, and four. Then the perfect points in the fourth order neighborhood are repeated on a magnified scale in order to show the fifth and sixth order neighborhoods in proper perspective.

TABLE 10. COMPLETE TABULATION OF PERFECT POINTS[10] IN THE FIRST SIX NEIGHBORHOODS, USING $x_1':x_3':x_4'=x_1:E^2x_3:E^3x_4$

P^3_3	P^2_{34}	P^7_{344}	P^{11}_{3444}	$P^{2,\ 3}_{34444}$
				$P^{3,\ 5}_{34441}$
			$P^{2,\ 5}_{3441}$	P^{13}_{34414}
				P^{17}_{34411}
		P^5_{341}	$P^{2,\ 3}_{3414}$	P^{11}_{34144}
				P^{7}_{34141}
			P^{3}_{3411}	P^{13}_{34114}
				$P^{2,\ 7}_{34111}$
	P^5_{31}	P^7_{314}	$P^{2,\ 3}_{3144}$	P^{19}_{31444}
				P^{17}_{31441}
			P^{3}_{3141}	P^{11}_{31414}
				P^{2}_{31411}
		P^2_{311}	P^{11}_{3114}	P^{19}_{31144}
				$P^{2,\ 7}_{31141}$
			P^{13}_{3111}	$P^{2,\ 3}_{31114}$
				$P^{3,\ 7}_{31111}$

[10] Occasionally two or more points are designated by a single notation. For example, $P^{2,\ 3}_{34444}$ represents the two points P^2_{34444} and P^3_{34444}

TABLE 10. (Continued)

$P^{2,\,3}_{34444}$	P^{29}_{344444}	$P^{47}_{3444444}$
		$P^{2,\,5}_{3444441}$
	P^{5}_{344441}	$P^{2}_{3444414}$
		$P^{43}_{3444411}$
$P^{3,\,5}_{34441}$	P^{19}_{344414}	$P^{2,\,17}_{3444144}$
		$P^{23}_{3444141}$
	$P^{2,\,13}_{344411}$	$P^{37}_{3444114}$
		$P^{41}_{3444111}$
P^{13}_{34414}	P^{23}_{344144}	$P^{2,\,3}_{3441444}$
		$P^{3,\,11}_{3441441}$
	P^{2}_{344141}	$P^{19}_{3441414}$
		$P^{29}_{3441411}$
P^{17}_{34411}	$P^{2,\,3}_{344114}$	$P^{41}_{3441144}$
		$P^{31}_{3441141}$
	P^{3}_{344111}	$P^{37}_{3441114}$
		$P^{2,\,11}_{3441111}$
P^{11}_{34144}	P^{17}_{341444}	$P^{2,\,7}_{3414444}$
		$P^{23}_{3414441}$
	P^{2}_{341441}	$P^{3,\,7}_{3414414}$
		$P^{3}_{3414411}$

TABLE 10. (Continued)

P^{7}_{34141}	P^{2}_{341414}	$P^{3,\ 5}_{3414144}$
		$P^{3}_{3414141}$
	P^{13}_{341411}	$P^{19}_{3414114}$
		$P^{2,\ 5}_{3414111}$
P^{13}_{34114}	$P^{2,\ 11}_{341144}$	$P^{5,\ 7}_{3411444}$
		$P^{31}_{3411441}$
	P^{17}_{341141}	$P^{3,\ 7}_{3411414}$
		$P^{2,\ 3,\ 5}_{3411411}$
$P^{2,\ 7}_{34111}$	P^{19}_{341114}	$P^{3,\ 11}_{3411144}$
		$P^{2,\ 3}_{3411141}$
	P^{23}_{341111}	$P^{2}_{3411114}$
		$P^{37}_{3411111}$
P^{19}_{31444}	P^{31}_{314444}	$P^{2,\ 5}_{3144444}$
		$P^{43}_{3144441}$
	$P^{2,\ 13}_{314441}$	$P^{3,\ 11}_{3144414}$
		$P^{3,\ 5}_{3144411}$
P^{17}_{31441}	$P^{2,\ 11}_{314414}$	$P^{3,\ 13}_{3144144}$
		$P^{3}_{3144141}$
	P^{29}_{314411}	$P^{41}_{3144114}$
		$P^{2,\ 23}_{3144111}$
P^{11}_{31414}	$P^{2,\ 5}_{314144}$	$P^{31}_{3141444}$
		$P^{29}_{3141441}$
	P^{13}_{314141}	$P^{3,\ 5}_{3141414}$
		$P^{2,\ 3}_{3141411}$

TABLE 10. (Continued)

P^{2}_{31411}	P^{23}_{314114}	$P^{3,\ 13}_{3141144}$
		$P^{2,\ 3,\ 5}_{3141141}$
	P^{5}_{314111}	$P^{2,\ 17}_{3141114}$
		$P^{41}_{3141111}$
P^{19}_{31144}	$P^{2,\ 3,\ 5}_{311444}$	$P^{7}_{3114444}$
		$P^{41}_{3114441}$
	P^{3}_{311441}	$P^{5,\ 7}_{3114414}$
		$P^{2,\ 23}_{3114411}$
$P^{2,7}_{31141}$	P^{17}_{311414}	$P^{31}_{3114144}$
		$P^{2,\ 5}_{3114141}$
	P^{5}_{311411}	$P^{2,\ 3}_{3114114}$
		$P^{3,\ 13}_{3114111}$
$P^{2,\ 3}_{31114}$	P^{31}_{311144}	$P^{7}_{3111444}$
		$P^{2,\ 11}_{3111441}$
	P^{23}_{311141}	$P^{2\ ,\ 7}_{3111414}$
		$P^{41}_{3111411}$
$P^{3,\ 7}_{31111}$	P^{29}_{311114}	$P^{2,\ 5}_{3111144}$
		$P^{37}_{3111141}$
	$P^{2,\ 17}_{311111}$	$P^{47}_{3111114}$
		$P^{5,\ 11}_{3111111}$

TABLE 11. BRIEF ARRANGEMENT OF THE FACTS IN TABLE 10.

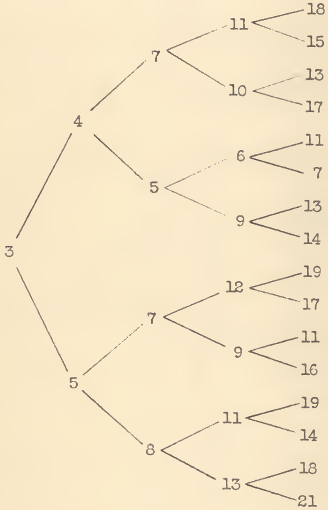

TABLE 11. (Continued)

18	29	47
		40
	25	32
		43
15	19	34
		23
	26	37
		41
13	23	36
		33
	16	19
		29
17	24	41
		31
	27	37
		44
11	17	28
		23
	16	21
		27

TABLE 11. (Continued)

7	8	15
		9
	13	19
		20
13	22	35
		31
	17	21
		30
14	19	33
		24
	23	32
		37
19	31	50
		43
	26	33
		45
17	22	39
		27
	29	41
		46
11	20	31
		29
	13	15
		24

TABLE 11. (Continued)

16	23	39
		30
	25	34
		41
19	30	49
		41
	27	35
		46
14	17	31
		20
	25	36
		39
18	31	49
		44
	23	28
		41
21	29	50
		37
	34	47
		55

10. The Hereditary Pattern.

In Table 11 the prime factors of the numbers appearing are the primes occurring in Table 10 with the same arrangement. At any given point in the r^{th} neighborhood (vertical column) the number a_r, occurring in the pattern, is determined by one of two recursive relationships. If the same quadratic transformation is used as was used to obtain the number preceding it in the chain, then simply add the two numbers a_{r-1} and a_{r-2}, in the $r\text{-}1^{st}$ and $r\text{-}2^{nd}$ neighborhoods (respectively) in this chain. If the same quadratic transformation is not used, then subtract the $r\text{-}2^{nd}$ number a_{r-2} from the $r\text{-}1^{st}$ number a_{r-1}, and add the result to the $r\text{-}1^{st}$ number in order to obtain the r^{th} number b_r. Or in brief,

$$a_r = a_{r-1} + a_{r-2} \quad \text{and}$$
$$b_r = 2a_{r-1} - a_{r-2}$$

While these formulas apply to Tables 10 and 11, the writer has used them in some other cases. It seems reasonable to assume that they may be extended to the more general case.

Some of the information in Table 10 is exhibited more clearly in Table 12, which centers attention upon the existence of perfect points.

TABLE 12. THE EXISTENCE OF PERFECT POINTS

Along the P_3P_4 direction							Along the P_3P_1 direction						
3	2	5	2	2	2	2	3	5	2	2	2	2	2
		7	3	3	3	3			7	3	3	3	3
			5	5	5	5				11	7	5	5
			11	7	11	7				13	11	13	7
				11	13	11					17	17	11
				13	17	17					19	23	13
				17	19	19						29	17
					23	23						31	23
					29	29						29	29
						31						31	31
						37							37
						41							41
						43							43
						47							47

In Table 12 the magnitude of a number indicates the period of the involution considered, whereas the

vertical column in which it lies tells us that a perfect point is in that neighborhood.

Thus, from Table 10 and Table 12 it is seen that for each prime considered in this study, a general theorem is true.

THEOREM 5. The imperfect point P_3 on F_4 has no perfect points in the neighborhoods of the 1^{st}, 2^{nd} ... j-1^{st} order under I_p (Each prime less than 47 determines a j). It does, however, have a perfect point in the j^{th} order neighborhood of P_3 along the invariant direction $P_3P_1(x_2=x_4=0)$. There is also a perfect point in the i^{th} order neighborhood along the direction $P_3P_4(x_1=x_2=0)$, where $i = j$ or $i = j+1$, and $j = 1, 2, 3, 4, 5, 6$. In particular,

if $p = 5$, then $i = 2$ and $j = 1$ at P^5_{341} and P^5_{31} respectively

if $p = 7$, then $i = 2$ and $j = 2$ at P^7_{344} and P^7_{314}

if $p = 11$, then $i = 3$ and $j = 3$ at P^{11}_{3444} and P^{11}_{3114}

if $p = 13$, then $i = 4$ and $j = 3$ at P^{13}_{34114} and P^{13}_{34414}

if $p = 17$, then $i = 4$ and $j = 4$ at P^{17}_{34414} and P^{17}_{31441}

if $p = 19$, then $i = 5$ and $j = 4$ at P^{19}_{344414} and P^{19}_{31444}

if $p = 23$, then $i = 5$ and $j = 5$ at P^{23}_{344144} and P^{23}_{314114}

if $p = 29$, then $i = 5$ and $j = 5$ at P^{29}_{344444} and P^{29}_{314411}

if $p = 31$, then $i = 6$ and $j = 5$ at $P^{31}_{3441141}$ and $P^{31}_{3141444}$

if $p = 37$, then $i = 6$ and $j = 6$ at $P^{37}_{3444114}$ and $P^{37}_{3111141}$

if $p = 41$, then $i = 6$ and $j = 6$ at $P^{41}_{3444111}$ and $P^{41}_{3144114}$

if $p = 43$, then $i = 6$ and $j = 6$ at $P^{43}_{3444411}$ and $P^{43}_{3144441}$

if $p = 47$, then $i = 6$ and $j = 6$ at $P^{47}_{3444444}$ and $P^{47}_{3111114}$

The points P^{19}_{31144}, P^{19}_{341114}, P^{23}_{341111}, P^{23}_{311141}, P^{29}_{311114}, $P^{31}_{3411441}$, $P^{37}_{3441114}$, $P^{37}_{3411111}$, $P^{41}_{3441144}$, $P^{41}_{3141111}$, $P^{41}_{3114441}$, $P^{41}_{3114441}$, and $P^{41}_{3111411}$ are also perfect points, with respect to the indicated involutions.

APPENDICES

Appendix A

A Systematic Arrangement of Exponents for the Invariant Polynomial Where p = 41

r = 0; n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20
b = 0,39,37,35,33,31,29,27,25,23,21,19,17,15,13,11, 9, 7, 5, 3, 1
c = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20

r = 1; n = 0, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20,21
b = 1, 38,36,34,32,30,28,26,24,22,20,18,16,14,12,10, 8, 6, 4, 2, 0
c = 39, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19

r = 2; n = 0, 1, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20,21
b = 2, 0, 35,33,31,29,27,25,23,21,19,17,15,13,11, 9, 7, 5, 3, 1
c = 37,38, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16,17

r = 3; n = 0, 1, 6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20,21,22
b = 3, 1, 32,30,28,26,24,22,20,18,16,14,12,10, 8, 6, 4, 2, 0
c = 35,36, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15,16

r = 4; n = 0, 1, 2, 8, 9,10,11,12,13,14,15,16,17,18,19,20,21,22
b = 4, 2, 0, 29,27,25,23,21,19,17,15,13,11, 9, 7, 5, 3, 1
c = 33,34,35, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14

r = 5; n = 0, 1, 2, 10,11,12,13,14,15,16,17,18,19,20,21,22,23
b = 5, 3, 1, 26,24,22,20,18,16,14,12,10, 8, 6, 4, 2, 0
c = 31,32,33, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13

r = 6; n = 0, 1, 2, 3, 12,13,14,15,16,17,18,19,20,21,22,23
b = 6, 4, 2, 0, 23,21,19,17,15,13,11, 9, 7, 5, 3, 1
c = 29,30,31,32, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10,11

Appendix A

A Systematic Arrangement of Exponents for the Invariant Polynomial (Continued)

r = 7;	n =	0, 1, 2, 3,	14,15,16,17,18,19,20,21,22,23,24
	b =	7, 5, 3, 1,	20,18,16,14,12,10, 8, 6, 4, 2, 0
	c =	27,28,29,30,	0, 1, 2, 3, 4, 5, 6, 7, 8, 9,10
r = 8;	n =	0, 1, 2, 3, 4,	16,17,18,19,20,21,22,23,24
	b =	8, 6, 4, 2, 0,	17,15,13,11, 9, 7, 5, 3, 1
	c =	25,26,27,28,29,	0, 1, 2, 3, 4, 5, 6, 7, 8
r = 9;	n =	0, 1, 2, 3, 4,	18,19,20,21,22,23,24,25
	b =	9, 7, 5, 3, 1,	14,12,10, 8, 6, 4, 2, 0
	c =	23,24,25,26,27,	0, 1, 2, 3, 4, 5, 6, 7
r =10;	n =	0, 1, 2, 3, 4, 5,	20,21,22,23,24,25
	b =	10, 8, 6, 4, 2, 0,	11, 9, 7, 5, 3, 1
	c =	21,22,23,24,25,26,	0, 1, 2, 3, 4, 5
r =11;	n =	0, 1, 2, 3, 4, 5,	22,23,24,25,26
	b =	11, 9, 7, 5, 3, 1,	8, 6, 4, 2, 0
	c =	19,20,21,22,23,24,	0, 1, 2, 3, 4
r =12;	n =	0, 1, 2, 3, 4, 5, 6,	24,25,26
	b =	12,10, 8, 6, 4, 2, 0,	5, 3, 1
	c =	17,18,19,20,21,22,23,	0, 1, 2
r =13;	n =	0, 1, 2, 3, 4, 5, 6,	26,27
	b =	13,11, 9, 7, 5, 3, 1,	2, 0
	c =	15,16,17,18,19,20,21,	0, 1

Appendix A

A Systematic Arrangement of Exponents for the Invariant Polynomial (Continued)

r = 14; n = 0, 1, 2, 3, 4, 5, 6, 7,
b = 14, 12, 10, 8, 6, 4, 2, 0,
c = 13, 14, 15, 16, 17, 18, 19, 20

r = 15; n = 0, 1, 2, 3, 4, 5, 6, 7,
b = 15, 13, 11, 9, 7, 5, 3, 1
c = 11, 12, 13, 14, 15, 16, 17, 18

r = 16; n = 0, 1, 2, 3, 4, 5, 6, 7, 8
b = 16, 14, 12, 10, 8, 6, 4, 2, 0
c = 9, 10, 11, 12, 13, 14, 15, 16, 17

r = 17; n = 0, 1, 2, 3, 4, 5, 6, 7, 8
b = 17, 15, 13, 11, 9, 7, 5, 3, 1
c = 7, 8, 9, 10, 11, 12, 13, 14, 15

r = 18; n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
b = 18, 16, 14, 12, 10, 8, 6, 4, 2, 0
c = 5, 6, 7, 8, 9, 10, 11, 12, 13, 14

r = 19; n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
b = 19, 17, 15, 13, 11, 9, 7, 5, 3, 1
c = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

r = 20; n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
b = 20, 18, 16, 14, 12, 10, 8, 6, 4, 2, 0
c = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

Appendix A

A Systematic Arrangement of Exponents for the Invariant Polynomial (Continued)

```
r = 21;  n =   1, 2, 3, 4, 5, 6, 7, 8, 9,10
         b =  19,17,15,13,11, 9, 7, 5, 3, 1
         c =   0, 1, 2, 3, 4, 5, 6, 7, 8, 9

r = 22;  n =         3, 4, 5, 6, 7, 8, 9,10,11
         b =        16,14,12,10, 8, 6, 4, 2, 0
         c =         0, 1, 2, 3, 4, 5, 6, 7, 8

r = 23;  n =               5, 6, 7, 8, 9,10,11
         b =              13,11, 9, 7, 5, 3, 1
         c =               0, 1, 2, 3, 4, 5, 6

r = 24;  n =                     7, 8, 9,10,11,12
         b =                    10, 8, 6, 4, 2, 0
         c =                     0, 1, 2, 3, 4, 5

r = 25;  n =                           9,10,11,12
         b =                           7, 5, 3, 1
         c =                           0, 1, 2, 3

r = 26;  n =                                11,12,13
         b =                                 4, 2, 0
         c =                                 0, 1, 2

r = 27;  n =                                      13
         b =                                       1
         c =                                       0
```

Appendix B

Specific Examples of the System of Counting the Number of Terms in the Most General Polynomial of Prime Degree p in x_1, x_2, x_3, and x_4, Invariant under T

For $p = 5$ in (A1) $S_n = 2+3+5+\ldots+\frac{p+1}{2} = 2+3 \qquad = 5$

in (A2) $S_n = 1+1+2+2+\ldots+\frac{p+3}{4} = 1+1+2 \qquad = 4$

in (A3) $S_n = 1+3+4+\ldots+\frac{p-1}{4} = 1 \qquad = \underline{1}$

$\qquad 10$

To obtain the total, the two terms x_1^p and x_4^p must be added $\qquad 10 + 2 \qquad = 12$

For easier reference, the above calculation, as well as succeeding examples will be written so as to more closely correspond to the form of Table 2. The series derived from (A1) will be shown in the first column, and those from (A2) and (A3) in the second column, with those from (A3) underlined.

Appendix B

Specific Examples (Continued)

For p = 5

3 + 1 = 4
2 + 1 = 3
2 =
1 = 1
5 + 5 = 10
Total = 10 + 2 = 12

For p = 7

4 + 1 = 5
3 + 1 = 4
1 + 2 = 3
2 = 2
2 = 2
8 + 8 = 16
Total = 16 + 2 = 18

For p = 11

6 + 1 = 7
5 + 1 = 6
3 + 2 = 5
2 + 2 = 4
3 = 3
3 = 3
3 = 3
1 = 1
16 +16 = 32
Total = 32 + 2 = 34

For p = 13

7 + 1 = 8
6 + 1 = 7
4 + 2 = 6
3 + 2 = 5
1 + 3 = 4
3 = 3
4 = 4
3 = 3
2 = 2
21 +21 = 42
Total = 42 + 2 = 44

For p = 17

9 + 1 = 10
8 + 1 = 9
6 + 2 = 8
5 + 2 = 7
3 + 3 = 6
2 + 3 = 5
4 = 4
4 = 4
5 = 5
4 = 4
3 = 3
1 = 1
33 +33 = 66
Total = 66 + 2 = 68

For p = 19

10 + 1 = 11
9 + 1 = 10
7 + 2 = 9
6 + 2 = 8
4 + 3 = 7
3 + 3 = 6
1 + 4 = 5
4 = 4
5 = 5
5 = 5
5 = 5
3 = 3
2 = 2
40 +40 = 80
Total = 80 + 2 = 82

Appendix B

Specific Examples (Continued)

For p = 23

```
12 + 1 = 13
11 + 1 = 12
 9 + 2 = 11
 8 + 2 = 10
 6 + 3 =  9
 5 + 3 =  8
 3 + 4 =  7
 2 + 4 =  6
     5 =  5
     5 =  5
     6 =  6
     6 =  6
     6 =  6
     4 =  4
     3 =  3
     1 =  1
56 +56 =112
 Total =112 + 2 = 114
```

For p = 29

```
15 + 1 = 16
14 + 1 = 15
12 + 2 = 14
11 + 2 = 13
 9 + 3 = 12
 8 + 3 = 11
 6 + 4 = 10
 5 + 4 =  9
 3 + 5 =  8
 2 + 5 =  7
     6 =  6
     6 =  6
     7 =  7
     7 =  7
     8 =  8
     7 =  7
     6 =  6
     4 =  4
     3 =  3
     1 =  1
85 +85 =170
 Total =170 + 2 = 172
```

Appendix B

Specific Examples (Continued)

For p = 31

```
16 + 1 = 17
15 + 1 = 16
13 + 2 = 15
12 + 2 = 14
10 + 3 = 13
 9 + 3 = 12
 7 + 4 = 11
 6 + 4 = 10
 4 + 5 =  9
 3 + 5 =  8
 1 + 6 =  7
     6 =  6
     7 =  7
     7 =  7
     8 =  8
     8 =  8
     8 =  8
     6 =  6
     5 =  5
     3 =  3
     2 =  2
----------
95 +95 =190
 Total =190 + 2 = 192
```

For p = 37

```
19 + 1 = 20
18 + 1 = 19
16 + 2 = 18
15 + 2 = 17
13 + 3 = 16
12 + 3 = 15
10 + 4 = 14
 9 + 4 = 13
 7 + 5 = 12
 6 + 5 = 11
 4 + 6 = 10
 3 + 6 =  9
 1 + 7 =  8
     7 =  7
     8 =  8
     8 =  8
     9 =  9
     9 =  9
    10 = 10
     9 =  9
     8 =  8
     6 =  6
     5 =  5
     3 =  3
     2 =  2
------------
133+133 =266
  Total =266 + 2 = 268
```

Appendix B

Specific Examples (Continued)

For p = 41

21 + 1 = 22
20 + 1 = 21
18 + 2 = 20
17 + 2 = 19
15 + 3 = 18
14 + 3 = 17
12 + 4 = 16
11 + 4 = 15
9 + 5 = 14
8 + 5 = 13
6 + 6 = 12
5 + 6 = 11
3 + 7 = 10
2 + 7 = 9
8 = 8
8 = 8
9 = 9
9 = 9
10 = 10
10 = 10
11 = 11
10 = 10
9 = 9
7 = 7
6 = 6
4 = 4
3 = 3
1 = 1
161+161 = 322
Total = 322 + 2 = 324

For p = 43

22 + 1 = 23
21 + 1 = 22
19 + 2 = 21
18 + 2 = 20
16 + 3 = 19
15 + 3 = 18
13 + 4 = 17
12 + 4 = 16
10 + 5 = 15
9 + 5 = 14
7 + 6 = 13
6 + 6 = 12
4 + 7 = 11
3 + 7 = 10
1 + 8 = 9
8 = 8
9 = 9
9 = 9
10 = 10
10 = 10
11 = 11
11 = 11
11 = 11
9 = 9
8 = 8
6 = 6
5 = 5
3 = 3
2 = 2
176+176 = 352
Total = 352 + 2 = 354

Appendix B

Specific Examples (Continued)

For p = 47

24 + 1 = 25
23 + 1 = 24
21 + 2 = 23
20 + 2 = 22
18 + 3 = 21
17 + 3 = 20
15 + 4 = 19
14 + 4 = 18
12 + 5 = 17
11 + 5 = 16
9 + 6 = 15
8 + 6 = 14
6 + 7 = 13
5 + 7 = 12
3 + 8 = 11
2 + 8 = 10
9 = 9
9 = 9
10 = 10
10 = 10
11 = 11
11 = 11
12 = 12
12 = 12
12 = 12
10 = 10
9 = 9
7 = 7
6 = 6
4 = 4
3 = 3
1 = 1
208+208 = 416
Total = 416 + 2 = 428

For p = 53

27 + 1 = 28
26 + 1 = 27
24 + 2 = 26
23 + 2 = 25
21 + 3 = 24
20 + 3 = 23
18 + 4 = 22
17 + 4 = 21
15 + 5 = 20
14 + 5 = 19
12 + 6 = 18
11 + 6 = 17
9 + 7 = 16
8 + 7 = 15
6 + 8 = 14
5 + 8 = 13
3 + 9 = 12
2 + 9 = 11
10 = 10
10 = 10
11 = 11
11 = 11
12 = 12
12 = 12
13 = 13
13 = 13
14 = 14
13 = 13
12 = 12
10 = 10
9 = 9
7 = 7
6 = 6
4 = 4
3 = 3
1 = 1
261+261 = 522
Total = 522 + 2 = 524

Appendix B

Specific Examples (Continued)

For p = 59

30 + 1	=	31
29 + 1	=	30
27 + 2	=	29
26 + 2	=	28
24 + 3	=	27
23 + 3	=	26
21 + 4	=	25
20 + 4	=	24
18 + 5	=	23
17 + 5	=	22
15 + 6	=	21
14 + 6	=	20
12 + 7	=	19
11 + 7	=	18
9 + 8	=	17
8 + 8	=	16
6 + 9	=	15
5 + 9	=	14
3 +10	=	13
2 +10	=	12
11	=	11
11	=	11
12	=	12
12	=	12
13	=	13
13	=	13
14	=	14
14	=	14
15	=	15
15	=	15
15	=	15
13	=	13
12	=	12
10	=	10
9	=	9
7	=	7
6	=	6
4	=	4
3	=	3
1	=	1
320+320	=	640

Total = 640 + 2 = 642

For p = 61

31 + 1	=	32
30 + 1	=	31
28 + 2	=	30
27 + 2	=	29
25 + 3	=	28
24 + 3	=	27
22 + 4	=	26
21 + 4	=	25
19 + 5	=	24
18 + 5	=	23
16 + 6	=	22
15 + 6	=	21
13 + 7	=	20
12 + 7	=	19
10 + 8	=	18
9 + 8	=	17
7 + 9	=	16
6 + 9	=	15
4 +10	=	14
3 +10	=	13
1 +11	=	12
11	=	11
12	=	12
12	=	12
13	=	13
13	=	13
14	=	14
14	=	14
15	=	15
15	=	15
16	=	16
15	=	15
14	=	14
12	=	12
11	=	11
9	=	9
8	=	8
6	=	6
5	=	5
3	=	3
2	=	2
341+341	=	682

Total = 682 + 2 = 684

Appendix C

A Systematic Arrangement of Exponents for the Most General Homogeneous Polynomial of Degree 13

The Invariant Polynomial

r = 0;	n =	0	1	2	3	4	5	6		
	b =	0	11	9	7	5	3	1		
	c =	0	1	2	3	4	5	6		
r = 1;	n =	0		2	3	4	5	6	7	
	b =	1		10	8	6	4	2	0	
	c =	11		0	1	2	3	4	5	
r = 2;	n =	0	1			4	5	6	7	
	b =	2	0			7	5	3	1	
	c =	9	10			0	1	2	3	
r = 3;	n =	0	1					6	7	8
	b =	3	1					4	2	0
	c =	7	8					0	1	2
r = 4;	n =	0	1	2						8
	b =	4	2	0						1
	c =	5	6	7						0
r = 5;	n =	0	1	2						
	b =	5	3	1						
	c =	3	4	5						
r = 6;	n =	0	1	2	3					
	b =	6	4	2	0					
	c =	1	2	3	4					
r = 7;	n =		1	2	3					
	b =		5	3	1					
	c =		0	1	2					
r = 8;	n =				3	4				
	b =				2	0				
	c =				0	1				

Appendix C

The Polynomial Containing E as a Factor When Transformed by T

r = 0;	n =	0,	1,	2,	3,	4,	5,	6,			12
	b =	12,	10,	8,	6,	4,	2,	0,			1
	c =	1,	2,	3,	4,	5,	6,	7,			0
r = 1;	n =	0,	1,	2,	3,	4,	5,	6			
	b =	0,	11,	9,	7,	5,	3,	1			
	c =	12,	0,	1,	2,	3,	4,	5			
r = 2;	n =	0,			3,	4,	5,	6,	7		
	b =	1,			8,	6,	4,	2,	0		
	c =	10,			0,	1,	2,	3,	4		
r = 3;	n =	0,	1,				5,	6,	7		
	b =	2,	0,				5,	3,	1		
	c =	8,	9,				0,	1,	2		
r = 4;	n =	0,	1,						7,	8	
	b =	3,	1,						2,	0	
	c =	6,	7,						0,	1	
r = 5;	n =	0,	1,	2							
	b =	4,	2,	0							
	c =	4,	5,	6							
r = 6;	n =	0,	1,	2							
	b =	5,	3,	1							
	c =	2,	3,	4							
r = 7;	n =	0,	1,	2,	3						
	b =	6,	4,	2,	0						
	c =	0,	1,	2,	3						
r = 8;	n =			2,	3						
	b =			3,	1						
	c =			0,	1						
r = 9;	n =					4					
	b =					0					
	c =					0					

Appendix C

The Polynomial Containing E^2 as a Factor When Transformed by T

r = 0;	n =	0	1	2	3	4	5				11	12
	b =	11	9	7	5	3	1				2	0
	c =	2	3	4	5	6	7				0	1
r = 1;	n =	0	1	2	3	4	5	6				
	b =	12	10	8	6	4	2	0				
	c =	0	1	2	3	4	5	6				
r = 2;	n =	0		2	3	4	5	6				
	b =	0		9	7	5	3	1				
	c =	11		0	1	2	3	4				
r = 3;	n =	0				4	5	6	7			
	b =	1				6	4	2	0			
	c =	9				0	1	2	3			
r = 4;	n =	0	1					6	7			
	b =	2	0					3	1			
	c =	7	8					0	1			
r = 5;	n =	0	1							8		
	b =	3	1							0		
	c =	5	6							0		
r = 6;	n =	0	1	2								
	b =	4	2	0								
	c =	3	4	5								
r = 7;	n =	0	1	2								
	b =	5	3	1								
	c =	1	2	3								
r = 8;	n =		1	2	3							
	b =		4	2	0							
	c =		0	1	2							
r = 9;	n =				3							
	b =				1							
	c =				0							

Appendix C

The Polynomial Containing E^3 as a Factor When Transformed by T

r = 0;	n =	0	1	2	3	4	5					10	11	
	b =	10	8	6	4	2	0					3	1	
	c =	3	4	5	6	7	8					0	1	
r = 1;	n =	0	1	2	3	4	5							12
	b =	11	9	7	5	3	1							0
	c =	1	2	3	4	5	6							0
r = 2;	n =		1	2	3	4	5	6						
	b =		10	8	6	4	2	0						
	c =		0	1	2	3	4	5						
r = 3;	n =	0			3	4	5	6						
	b =	0			7	5	3	1						
	c =	10			0	1	2	3						
r = 4;	n =	0					5	6	7					
	b =	1					4	2	0					
	c =	8					0	1	2					
r = 5;	n =	0	1						7					
	b =	2	0						1					
	c =	6	7						0					
r = 6;	n =	0	1											
	b =	3	1											
	c =	4	5											
r = 7;	n =	0	1	2										
	b =	4	2	0										
	c =	2	3	4										
r = 8;	n =	0	1	2										
	b =	5	3	1										
	c =	0	1	2										
r = 9;	n =			2	3									
	b =			2	0									
	c =			0	1									

Appendix C

The Polynomial Containing E^4 as a Factor When Transformed by T

r = 0;	n =	0,	1,	2,	3,	4,				9,	10,	11
	b =	9,	7,	5,	3,	1,				4,	2,	0
	c =	4,	5,	6,	7,	8,				0,	1,	2
r = 1;	n =	0,	1,	2,	3,	4,	5,					11
	b =	10,	8,	6,	4,	2,	0,					1
	c =	2,	3,	4,	5,	6,	7,					0
r = 2;	n =	0,	1,	2,	3,	4,	5					
	b =	11,	9,	7,	5,	3,	1					
	c =	0,	1,	2,	3,	4,	5					
r = 3;	n =			2,	3,	4,	5,	6				
	b =			8,	6,	4,	2,	0				
	c =			0,	1,	2,	3,	4				
r = 4;	n =	0,				4,	5,	6				
	b =	0,				5,	3,	1				
	c =	9,				0,	1,	2				
r = 5;	n =	0,						6,	7			
	b =	1,						2,	0			
	c =	7,						0,	1			
r = 6;	n =	0,	1									
	b =	2,	0									
	c =	5,	6									
r = 7;	n =	0,	1									
	b =	3,	1									
	c =	3,	4									
r = 8;	n =	0,	1,	2								
	b =	4,	2,	0								
	c =	1,	2,	3								
r = 9;	n =		1,	2								
	b =		3,	1								
	c =		0,	1								
r =10;	n =				3							
	b =				0							
	c =				0							

Appendix C

The Polynomial Containing E^5 as a Factor When Transformed by T

r = 0;	n =	0	1	2	3	4				8	9	10	
	b =	8	6	4	2	0				5	3	1	
	c =	5	6	7	8	9				0	1	2	
r = 1;	n =	0	1	2	3	4						10	11
	b =	9	7	5	3	1						2	0
	c =	3	4	5	6	7						0	1
r = 2;	n =	0	1	2	3	4	5						
	b =	10	8	6	4	2	0						
	c =	1	2	3	4	5	6						
r = 3;	n =		1	2	3	4	5						
	b =		9	7	5	3	1						
	c =		0	1	2	3	4						
r = 4;	n =				3	4	5	6					
	b =				6	4	2	0					
	c =				0	1	2	3					
r = 5;	n =	0					5	6					
	b =	0					3	1					
	c =	8					0	1					
r = 6;	n =	0							7				
	b =	1							0				
	c =	6							0				
r = 7;	n =	0	1										
	b =	2	0										
	c =	4	5										
r = 8;	n =	0	1										
	b =	3	1										
	c =	2	3										
r = 9;	n =	0	1	2									
	b =	4	2	0									
	c =	0	1	2									
r = 10;	n =			2									
	b =			1									
	c =			0									

Appendix C

The Polynomial Containing E^6 as a Factor When Transformed by T

r = 0;	n =	0	1	2	3				7	8	9	10	
	b =	7	5	3	1				6	4	2	0	
	c =	6	7	8	9				0	1	2	3	
r = 1;	n =	0	1	2	3	4					9	10	
	b =	8	6	4	2	0					3	1	
	c =	4	5	6	7	8					0	1	
r = 2;	n =	0	1	2	3	4							11
	b =	9	7	5	3	1							0
	c =	2	3	4	5	6							0
r = 3;	n =	0	1	2	3	4	5						
	b =	10	8	6	4	2	0						
	c =	0	1	2	3	4	5						
r = 4;	n =			2	3	4	5						
	b =			7	5	3	1						
	c =			0	1	2	3						
r = 5;	n =					4	5	6					
	b =					4	2	0					
	c =					0	1	2					
r = 6;	n =	0						6					
	b =	0						1					
	c =	7						0					
r = 7;	n =	0											
	b =	1											
	c =	5											
r = 8;	n =	0	1										
	b =	2	0										
	c =	3	4										
r = 9;	n =	0	1										
	b =	3	1										
	c =	1	2										
r = 10;	n =		1	2									
	b =		2	0									
	c =		0	1									

Appendix C

The Polynomial Containing E^7 as a Factor When Transformed by T

r = 0;	n =	0	1	2	3		6	7	8	9	
	b =	6	4	2	0		7	5	3	1	
	c =	7	8	9	10		0	1	2	3	
r = 1;	n =	0	1	2	3				8	9	10
	b =	7	5	3	1				4	2	0
	c =	5	6	7	8				0	1	2
r = 2;	n =	0	1	2	3	4					10
	b =	8	6	4	2	0					1
	c =	3	4	5	6	7					0
r = 3;	n =	0	1	2	3	4					
	b =	9	7	5	3	1					
	c =	1	2	3	4	5					
r = 4;	n =		1	2	3	4	5				
	b =		8	6	4	2	0				
	c =		0	1	2	3	4				
r = 5;	n =				3	4	5				
	b =				5	3	1				
	c =				0	1	2				
r = 6;	n =						5	6			
	b =						2	0			
	c =						0	1			
r = 7;	n =	0									
	b =	0									
	c =	6									
r = 8;	n =	0									
	b =	1									
	c =	4									
r = 9;	n =	0	1								
	b =	2	0								
	c =	2	3								
r =10;	n =	0	1								
	b =	3	1								
	c =	0	1								
r =11;	n =			2							
	b =			0							
	c =			0							

Appendix C

The Polynomial Containing E^8 as a Factor When Transformed by T

r = 0;	n =	0, 1, 2, 5, 6, 7, 8, 9
	b =	5, 3, 1, 8, 6, 4, 2, 0
	c =	8, 9, 10, 0, 1, 2, 3, 4
r = 1;	n =	0, 1, 2, 3, 7, 8, 9
	b =	6, 4, 2, 0, 5, 3, 1
	c =	6, 7, 8, 9, 0, 1, 2
r = 2;	n =	0, 1, 2, 3, 9, 10
	b =	7, 5, 3, 1, 2, 0
	c =	4, 5, 6, 7, 0, 1
r = 3;	n =	0, 1, 2, 3, 4
	b =	8, 6, 4, 2, 0
	c =	2, 3, 4, 5, 6
r = 4;	n =	0, 1, 2, 3, 4
	b =	9, 7, 5, 3, 1
	c =	0, 1, 2, 3, 4
r = 5;	n =	2, 3, 4, 5
	b =	6, 4, 2, 0
	c =	0, 1, 2, 3
r = 6;	n =	4, 5
	b =	3, 1
	c =	0, 1
r = 7;	n =	6
	b =	0
	c =	0
r = 8;	n =	0
	b =	0
	c =	5
r = 9;	n =	0
	b =	1
	c =	3
r = 10;	n =	0, 1
	b =	2, 0
	c =	1, 2
r = 11;	n =	1
	b =	1
	c =	0

Appendix C

The Polynomial Containing E^9 as a Factor When Transformed by T

r = 0;	n =	0	1	2		4	5	6	7	8		
	b =	4	2	0		9	7	5	3	1		
	c =	9	10	11		0	1	2	3	4		
r = 1;	n =	0	1	2				6	7	8	9	
	b =	5	3	1				6	4	2	0	
	c =	7	8	9				0	1	2	3	
r = 2;	n =	0	1	2	3					8	9	
	b =	6	4	2	0					3	1	
	c =	5	6	7	8					0	1	
r = 3;	n =	0	1	2	3							10
	b =	7	5	3	1							0
	c =	3	4	5	6							0
r = 4;	n =	0	1	2	3	4						
	b =	8	6	4	2	0						
	c =	1	2	3	4	5						
r = 5;	n =		1	2	3	4						
	b =		7	5	3	1						
	c =		0	1	2	3						
r = 6;	n =				3	4	5					
	b =				4	2	0					
	c =				0	1	2					
r = 7;	n =						5					
	b =						1					
	c =						0					
r = 8;	No terms											
r = 9;	n =	0										
	b =	0										
	c =	4										
r = 10;	n =	0										
	b =	1										
	c =	2										
r = 11;	n =	0	1									
	b =	2	0									
	c =	0	1									

Appendix C

The Polynomial Containing E^{10} as a Factor When Transformed by T

r = 0;	n =	0	1		3	4	5	6	7	8	
	b =	3	1		10	8	6	4	2	0	
	c =	10	11		0	1	2	3	4	5	
r = 1;	n =	0	1	2			5	6	7	8	
	b =	4	2	0			7	5	3	1	
	c =	8	9	10			0	1	2	3	
r = 2;	n =	0	1	2					7	8	9
	b =	5	3	1					4	2	0
	c =	6	7	8					0	1	2
r = 3;	n =	0	1	2	3						9
	b =	6	4	2	0						1
	c =	4	5	6	7						0
r = 4;	n =	0	1	2	3						
	b =	7	5	3	1						
	c =	2	3	4	5						
r = 5;	n =	0	1	2	3	4					
	b =	8	6	4	2	0					
	c =	0	1	2	3	4					
r = 6;	n =			2	3	4					
	b =			5	3	1					
	c =			0	1	2					
r = 7;	n =					4	5				
	b =					2	0				
	c =					0	1				
r = 8,9	No terms										
r =10;	n =	0									
	b =	0									
	c =	3									
r =11;	n =	0									
	b =	1									
	c =	1									
r =12;	n =		1								
	b =		0								
	c =		0								

Appendix C

The Polynomial Containing E^{11} as a Factor When Transformed by T

r = 0;	n =	0,	1,	2,	3,	4,	5,	6,	7		
	b =	2,	0,	11,	9,	7,	5,	3,	1		
	c =	11,	12,	0,	1,	2,	3,	4,	5		
r = 1;	n =	0,	1,			4,	5,	6,	7,	8	
	b =	3,	1,			8,	6,	4,	2,	0	
	c =	9,	10,			0,	1,	2,	3,	4	
r = 2;	n =	0,	1,	2,				6,	7,	8	
	b =	4,	2,	0,				5,	3,	1	
	c =	7,	8,	9,				0,	1,	2	
r = 3;	n =	0,	1,	2,						8,	9
	b =	5,	3,	1,						2,	0
	c =	5,	6,	7,						0,	1
r = 4;	n =	0,	1,	2,	3						
	b =	6,	4,	2,	0						
	c =	3,	4,	5,	6						
r = 5;	n =	0,	1,	2,	3						
	b =	7,	5,	3,	1						
	c =	1,	2,	3,	4						
r = 6;	n =		1,	2,	3,	4					
	b =		6,	4,	2,	0					
	c =		0,	1,	2,	3					
r = 7;	n =				3,	4					
	b =				3,	1					
	c =				0,	1					
r = 8;	n =						5				
	b =						0				
	c =						0				
r = 9, 10	No terms										
r = 11;	n =	0									
	b =	0									
	c =	2									
r = 12;	n =	0									
	b =	1									
	c =	0									

Appendix C

The Polynomial Containing E^{12} as a Factor When Transformed by T

r = 0;	n =	0	1	2	3	4	5	6	7		
	b =	1	12	10	8	6	4	2	0		
	c =	12	0	1	2	3	4	5	6		
r = 1;	n =	0	1		3	4	5	6	7		
	b =	2	0		9	7	5	3	1		
	c =	10	11		0	1	2	3	4		
r = 2;	n =	0	1				5	6	7	8	
	b =	3	1				6	4	2	0	
	c =	8	9				0	1	2	3	
r = 3;	n =	0	1	2					7	8	
	b =	4	2	0					3	1	
	c =	6	7	8					0	1	
r = 4;	n =	0	1	2							9
	b =	5	3	1							0
	c =	4	5	6							0
r = 5;	n =	0	1	2	3						
	b =	6	4	2	0						
	c =	2	3	4	5						
r = 6;	n =	0	1	2	3						
	b =	7	5	3	1						
	c =	0	1	2	3						
r = 7;	n =			2	3	4					
	b =			4	2	0					
	c =			0	1	2					
r = 8;	n =					4					
	b =					1					
	c =					0					
r = 9,10,11	No terms										
r = 12;	n =	0									
	b =	0									
	c =	1									

Appendix D

A Systematic Arrangement of Exponents for the Most General Homogeneous Polynomial of Degree 11

The Invariant Polynomial

r = 0;	n =	0	1	2	3	4	5		
	b =	0	9	7	5	3	1		
	c =	0	1	2	3	4	5		
r = 1;	n =	0		2	3	4	5	6	
	b =	1		8	6	4	2	0	
	c =	9		0	1	2	3	4	
r = 2;	n =	0	1			4	5	6	
	b =	2	0			5	3	1	
	c =	7	8			0	1	2	
r = 3;	n =	0	1					6	7
	b =	3	1					2	0
	c =	5	6					0	1
r = 4;	n =	0	1	2					
	b =	4	2	0					
	c =	3	4	5					
r = 5;	n =	0	1	2					
	b =	5	3	1					
	c =	1	2	3					
r = 6;	n =		1	2	3				
	b =		4	2	0				
	c =		0	1	2				
r = 7;	n =				3				
	b =				1				
	c =				0				

Appendix D

The Polynomial Containing E^s as a Factor When Transformed by T

$s = 1$

r = 0;	n =	0,	1,	2,	3,	4,	5,			10
	b =	10,	8,	6,	4,	2,	0,			1
	c =	1,	2,	3,	4,	5,	6,			0
r = 1;	n =	0,	1,	2,	3,	4,	5			
	b =	0,	9,	7,	5,	3,	1			
	c =	10,	0,	1,	2,	3,	4			
r = 2;	n =	0,			3,	4,	5,	6		
	b =	1,			6,	4,	2,	0		
	c =	8,			0,	1,	2,	3		
r = 3;	n =	0,	1,				5,	6		
	b =	2,	0,				3,	1		
	c =	6,	7,				0,	1		
r = 4;	n =	0,	1,						7	
	b =	3,	1,						0	
	c =	4,	5,						0	
r = 5;	n =	0,	1,	2						
	b =	4,	2,	0						
	c =	2,	3,	4						
r = 6;	n =	0,	1,	2						
	b =	5,	3,	1						
	c =	0,	1,	2						
r = 7;	n =			2,	3					
	b =			2,	0					
	c =			0,	1					

Appendix D

The Polynomial Containing E^s as a Factor When Transformed by T (Continued)
$s = 2$

r = 0;	n =	0	1	2	3	4			9	10
	b =	9	7	5	3	1			2	0
	c =	2	3	4	5	6			0	1
r = 1;	n =	0	1	2	3	4	5			
	b =	10	8	6	4	2	0			
	c =	0	1	2	3	4	5			
r = 2;	n =	0		2	3	4	5			
	b =	0		7	5	3	1			
	c =	9		0	1	2	3			
r = 3;	n =	0				4	5	6		
	b =	1				4	2	0		
	c =	7				0	1	2		
r = 4;	n =	0	1					6		
	b =	2	0					1		
	c =	5	6					0		
r = 5;	n =	0	1							
	b =	3	1							
	c =	3	4							
r = 6;	n =	0	1	2						
	b =	4	2	0						
	c =	1	2	3						
r = 7;	n =		1	2						
	b =		3	1						
	c =		0	1						
r = 8;	n =				3					
	b =				0					
	c =				0					

Appendix D

The Polynomial Containing E^s as a Factor When Transformed by T (Continued)

$s = 3$

r = 0;	n =	0,	1,	2,	3,	4,				8,	9	
	b =	8,	6,	4,	2,	0,				3,	1	
	c =	3,	4,	5,	6,	7,				0,	1	
r = 1;	n =	0,	1,	2,	3,	4,						10
	b =	9,	7,	5,	3,	1,						0
	c =	1,	2,	3,	4,	5,						0
r = 2;	n =		1,	2,	3,	4,	5					
	b =		8,	6,	4,	2,	0					
	c =		0,	1,	2,	3,	4					
r = 3;	n =	0,			3,	4,	5					
	b =	0,			5,	3,	1					
	c =	8,			0,	1,	2					
r = 4;	n =	0,					5,	6				
	b =	1,					2,	0				
	c =	6,					0,	1				
r = 5;	n =	0,	1									
	b =	2,	0									
	c =	4,	5									
r = 6;	n =	0,	1									
	b =	3,	1									
	c =	2,	3									
r = 7;	n =	0,	1,	2								
	b =	4,	2,	0								
	c =	0,	1,	2								
r = 8;	n =			2								
	b =			1								
	c =			0								

Appendix D

The Polynomial Containing E^s as a Factor When Transformed by T (Continued)

$s = 4$

r = 0;	n =	0,	1,	2,	3,				7,	8,	9
	b =	7,	5,	3,	1,				4,	2,	0
	c =	4,	5,	6,	7,				0,	1,	2
r = 1;	n =	0,	1,	2,	3,	4,					9
	b =	8,	6,	4,	2,	0,					1
	c =	2,	3,	4,	5,	6,					0
r = 2;	n =	0,	1,	2,	3,	4					
	b =	9,	7,	5,	3,	1					
	c =	0,	1,	2,	3,	4					
r = 3;	n =			2,	3,	4,	5				
	b =			6,	4,	2,	0				
	c =			0,	1,	2,	3				
r = 4;	n =	0,				4,	5				
	b =	0,				3,	1				
	c =	7,				0,	1				
r = 5;	n =	0,						6			
	b =	1,						0			
	c =	5,						0			
r = 6;	n =	0,	1								
	b =	2,	0								
	c =	3,	4								
r = 7;	n =	0,	1								
	b =	3,	1								
	c =	1,	2								
r = 8;	n =		1,	2							
	b =		2,	0							
	c =		0,	1							

Appendix D

The Polynomial Containing E^s as a Factor
When Transformed by T (Continued)
$s = 5$

r = 0;	n =	0	1	2	3			6	7	8	
	b =	6	4	2	0			5	3	1	
	c =	5	6	7	8			0	1	2	
r = 1;	n =	0	1	2	3					8	9
	b =	7	5	3	1					2	0
	c =	3	4	5	6					0	1
r = 2;	n =	0	1	2	3	4					
	b =	8	6	4	2	0					
	c =	1	2	3	4	5					
r = 3;	n =		1	2	3	4					
	b =		7	5	3	1					
	c =		0	1	2	3					
r = 4;	n =				3	4	5				
	b =				4	2	0				
	c =				0	1	2				
r = 5:	n =	0					5				
	b =	0					1				
	c =	6					0				
r = 6;	n =	0									
	b =	1									
	c =	4									
r = 7;	n =	0	1								
	b =	2	0								
	c =	2	3								
r = 8;	n =	0	1								
	b =	3	1								
	c =	0	1								
r = 9;	n =			2							
	b =			0							
	c =			0							

Appendix D

The Polynomial Containing E^s as a Factor When Transformed by T (Continued)

$s = 6$

```
r = 0;  n =  0, 1, 2,         5, 6, 7, 8
        b =  5, 3, 1,         6, 4, 2, 0
        c =  6, 7, 8,         0, 1, 2, 3

r = 1;  n =  0, 1, 2, 3,            7, 8
        b =  6, 4, 2, 0,            3, 1
        c =  4, 5, 6, 7,            0, 1

r = 2;  n =  0, 1, 2, 3,                  9
        b =  7, 5, 3, 1,                  0
        c =  2, 3, 4, 5,                  0

r = 3;  n =  0, 1, 2, 3, 4
        b =  8, 6, 4, 2, 0
        c =  0, 1, 2, 3, 4

r = 4;  n =        2, 3, 4
        b =        5, 3, 1
        c =        0, 1, 2

r = 5;  n =              4, 5
        b =              2, 0
        c =              0, 1

r = 6;  n =  0
        b =  0
        c =  5

r = 7;  n =  0
        b =  1
        c =  3

r = 8;  n =  0, 1
        b =  2, 0
        c =  1, 2

r = 9;  n =     1
        b =     1
        c =     0
```

Appendix D

The Polynomial Containing E^s as a Factor When Transformed by T (Continued)

$s = 7$

r = 0;	n =	0, 1, 2,	4, 5, 6, 7
	b =	4, 2, 0,	7, 5, 3, 1
	c =	7, 8, 9,	0, 1, 2, 3
r = 1;	n =	0, 1, 2,	6, 7, 8
	b =	5, 3, 1,	4, 2, 0
	c =	5, 6, 7,	0, 1, 2
r = 2;	n =	0, 1, 2, 3,	8
	b =	6, 4, 2, 0,	1
	c =	3, 4, 5, 6,	0
r = 3;	n =	0, 1, 2, 3	
	b =	7, 5, 3, 1	
	c =	1, 2, 3, 4	
r = 4;	n =	1, 2, 3, 4	
	b =	6, 4, 2, 0	
	c =	0, 1, 2, 3	
r = 5;	n =	3, 4	
	b =	3, 1	
	c =	0, 1	
r = 6;	n =	5	
	b =	0	
	c =	0	
r = 7;	n =	0	
	b =	0	
	c =	4	
r = 8;	n =	0	
	b =	1	
	c =	2	
r = 9;	n =	0, 1	
	b =	2, 0	
	c =	0, 1	

Appendix D

The Polynomial Containing E^s as a Factor When Transformed by T (Continued)

$s = 8$

r = 0;	n =	0,	1,		3,	4,	5,	6,	7	
	b =	3,	1,		8,	6,	4,	2,	0	
	c =	8,	9,		0,	1,	2,	3,	4	
r = 1;	n =	0,	1,	2,			5,	6,	7	
	b =	4,	2,	0,			5,	3,	1	
	c =	6,	7,	8,			0,	1,	2	
r = 2;	n =	0,	1,	2,					7,	8
	b =	5,	3,	1,					2,	0
	c =	4,	5,	6,					0,	1
r = 3;	n =	0,	1,	2,	3					
	b =	6,	4,	2,	0					
	c =	2,	3,	4,	5					
r = 4;	n =	0,	1,	2,	3					
	b =	7,	5,	3,	1					
	c =	0,	1,	2,	3					
r = 5;	n =			2,	3,	4				
	b =			4,	2,	0				
	c =			0,	1,	2				
r = 6;	n =					4				
	b =					1				
	c =					0				
r = 7;	No terms									
r = 8;	n =	0								
	b =	0								
	c =	3								
r = 9;	n =	0								
	b =	1								
	c =	1								
r =10;	n =		1							
	b =		0							
	c =		0							

Appendix D

The Polynomial Containing E^s as a Factor When Transformed by T (Continued)

$s = 9$

r = 0;	n =	0,	1,	2,	3,	4,	5,	6		
	b =	2,	0,	9,	7,	5,	3,	1		
	c =	9,	10,	0,	1,	2,	3,	4		
r = 1;	n =	0,	1,			4,	5,	6,	7	
	b =	3,	1,			6,	4,	2,	0	
	c =	7,	8,			0,	1,	2,	3	
r = 2;	n =	0,	1,	2,				6,	7	
	b =	4,	2,	0,				3,	1	
	c =	5,	6,	7,				0,	1	
r = 3;	n =	0,	1,	2,						8
	b =	5,	3,	1,						0
	c =	3,	4,	5,						0
r = 4;	n =	0,	1,	2,	3					
	b =	6,	4,	2,	0					
	c =	1,	2,	3,	4					
r = 5;	n =		1,	2,	3					
	b =		5,	3,	1					
	c =		0,	1,	2					
r = 6;	n =				3,	4				
	b =				2,	0				
	c =				0,	1				
r = 7,8 No terms										
r = 9;	n =	0								
	b =	0								
	c =	2								
r =10;	n =	0								
	b =	1								
	c =	0								

Appendix D

The Polynomial Containing E^s as a Factor When Transformed by T (Continued)

$s = 10$

r = 0;	n =	0,	1,	2,	3,	4,	5,	6	
	b =	1,	10,	8,	6,	4,	2,	0	
	c =	10,	0,	1,	2,	3,	4,	5	
r = 1;	n =	0,	1,		3,	4,	5,	6	
	b =	2,	0,		7,	5,	3,	1	
	c =	8,	9,		0,	1,	2,	3	
r = 2;	n =	0,	1,				5,	6,	7
	b =	3,	1,				4,	2,	0
	c =	6,	7,				0,	1,	2
r = 3;	n =	0,	1,	2,					7
	b =	4,	2,	0,					1
	c =	4,	5,	6,					0
r = 4;	n =	0,	1,	2					
	b =	5,	3,	1					
	c =	2,	3,	4					
r = 5;	n =	0,	1,	2,	3				
	b =	6,	4,	2,	0				
	c =	0,	1,	2,	3				
r = 6;	n =			2,	3				
	b =			3,	1				
	c =			0,	1				
r = 7;	n =					4			
	b =					0			
	c =					0			

r = 8,9; No terms

r =10;	n =	0
	b =	0
	c =	1

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BIOGRAPHICAL SKETCH

James Crutchfield Morelock was born in Martin, Tennessee on February 7, 1920. He attended high school in Fulton, Kentucky. After receiving the Degree of Bachelor of Science from Memphis State in 1941, he entered the Army Air Corps and served for four years. Upon release he attended the University of Missouri and received the degree of Master of Arts in mathematics. While at Missouri he was employed as Assistant Instructor in mathematics. Upon graduation he accepted a position as Interim Instructor at the University of Florida. He is at present a graduate Fellow in mathematics.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

August 16, 1952

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